Mathematical Models and Methods in Applied Sciences


# ON GEOMETRICAL RIGIDITY OF SURFACES. APPLICATION TO THE THEORY OF THIN LINEAR ELASTIC SHELLS

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In this paper we first present a panorama about geometrical rigidity and inextensional displacements (also called infinitesimal bendings) for surfaces with kinematic boundary conditions and for surfaces with edges (in the sense of folds or junctions). This theory is fundamental for thin linear elastic shells, as it rules their asymptotic behavior when the thickness tends to zero. This behavior enlights some difficulties encountered in numerical studies of very thin elastic shells. Our approach is based on the introduction of a non-classical space denoted by $\mathbf{R}(S)$ and related with inextensional displacements. It permits us to obtain new results concerning developable surfaces and hyperbolic surfaces, with one or two edges (most supposed to keep constant angle), including a theorem of rigid edge when the edge is an asymptotic line of the surface. By applying these results, we are able to exhibit a new example of sensitive problem for a shell with hyperbolic mean surface and with two edges keeping constant angle. In the appendix, we give a non-classical variant of Goursat problem for hyperbolic linear partial differential equations system, used in the proof of a rigidity result.

# 1 Introduction

A thin elastic shell is an elastic body, the shape of which is nearly a surface. It is deformed under the action of external forces supposed to be small in order to have small deformation and to *linearize* with respect to the displacement.

The study of an elastic shell, in particular its modeling by finite element methods, poses some difficult problems when the thickness of the shell is "very small". These difficulties are enlighted by the asymptotic behavior of a very thin elastic shell, where the notion of geometrical rigidity plays a fundamental role.

It is known that the asymptotic behavior of a linear thin elastic shell, when the thickness tends toward zero, depends essentially on the ability of the mean surface of the shell to deform in an inextensional way, see [29] or [30]. When this is not possible, one says that the mean surface is *geometrically rigid* or equivalently that the shell is *inhibited*.

All along this paper, the notion of rigididity is understood in its geometrical sense and is independent of the material or model of shell considered. We emphasize on this geometrical sense since rigidity is equivocal in a mechanical framework.

In the non-inhibited shell case (bending dominated case), the displacement field is inextensional and the shell presents some great weakness under applied external forces. This weakness can go unnoticed in numerical studies, see [1], [27], [7] and [10]. This is a phenomenon referred as *membrane locking*.

Among the inhibited shell (or membrane dominated shell), one distinguishes the *well-inhibited* shells, for which the limit problem is well-posed in a nice functional space, whereas in the not well inhibited case, the solution can be very sensitive with respect to the external forces, giving a so-called *sensitive problem* in the sense of Lions and Sanchez-Palencia [22], which can be seen as an instability phenomenon, see [20].

Actually, in the framework of Koiter's linear model of thin elastic shells, a modeling by finite elements methods exists and "works", see [3] for instance. But, in order to obtain a good approximation, it appears that the mesh step of the numerical scheme has to be, in some cases, as *small* as the thickness of the shell. A possible reason of a numerical locking arises here: in applications, since computers are limited in speed and memory, the mesh step is sometime taken too big for "very thin" elastic shells.

Thereby, the numerical difficulties we evocate indicates that a theoretical study of the geometrical rigidity of the mean surface of the shell is then necessary in order to have a good knowledge of its mechanical behavior.

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The theory of inextensional displacements and geometrical rigidity of a surface is classical. Many results about the subject has been compiled in celebrated *"Leçons sur la théorie générale des surfaces"* by G. Darboux, first published in 1894, [14].

Inextensional displacements are displacements that keep unchanged the intrinsic length of the surface (in the linearized sense); they satisfy a linear first order partial differential equations (P.D.E.) system in two variables we call *bending system* (it is also called *rigidity system*). The fundamental property of the bending system, is that its "*characteristics*"*coincide with the *asymptotic lines* of the surface.

About geometrical rigidity, the classical theory concerns essentially the elliptic surfaces without boundary (ovoids) and more generally the convex surfaces without any flat point. It is known that such surfaces are geometrically rigid, see [28] or [35].

Concerning the cases of surface with boundary, the rigidifying effects of kinematic boundary conditions such as fixation or clamping are classical, see [35], but for sake of completeness, we recall them in section 5.

Actually, the monograph "Generalized Analytic Functions" by I.N. Vekua, [35], contains many results of rigidity concerning surfaces without boundary such as ovoids and surfaces of revolution. Vekua also proved some rigidity result of elliptic

*In fact, it is the characteristics of a reduced form of the bending system, which can be expressed in various local coordinates such as Cartesian coordinates or covariant coordinates.

surfaces with edges, taking advantage of the theory of generalized analytic functions (also called "pseudo-analytic functions" by L.Bers [5]). But, this aspect of the theory of inextensional displacement is specific to elliptic surfaces.

A surface with an edge can be considered as a surface constituted of two smooth parts joined together along a curve. We assume all along this paper that along the edge , the respective tangent planes of the two adjacent smooth parts do not coincide and make an angle different from zero and $\pi$. We shall consider more particularily the case of edges keeping the angle of edge invariant during the deformation (we shall say *keeping constant angle*). The keeping constant angle hypothesis can be justified, in the mechanical point of view, for shells that are made of one piece, contrary to the shells that are folded, i.e. for which the elasticity property are damaged on the fold. It has been proved in the case of plates by H. Le Dret [19], see also [32] for some heuristic comments.

Up to our knowledge, it seems that no significant research have been made on cases of hyperbolic or developable surface with one or several edges, where edges are understood as folds or junctions of surfaces. In this direction, we found some new results.

Our approach of the theory is based on a non-classical space denoted $\mathbf{R}(S)$ developed in section 4. Introduced by the author in [8], the space $\mathbf{R}(S)$ is isomorphic to the space of inextensional displacement quotiented by the subspace of rigid displacements (also called trivial bendings). The space $\mathbf{R}(S)$ will prove to be the technical *key* to the proof of our new results of rigidity of surface with one or two edges we exhibit in sections 6, 7, 8 and 9.

Actually, a fundamental tangential property of the elements of $\mathbf{R}(S)$ (proposition 4.1) allow us to describe in a practical way the condition of continuity and keeping constant angle along an edge (proposition 6.3), which becomes somewhat complex when expressed in usual covariant components of a displacement.

We study then (section 7) various surfaces with an edge. We exhibit some surfaces rigidified by an edge, some non-rigid surfaces with an edge, in a somehow systematical study.

One of the new result of this paper is given in section 8. In the particular case of an edge keeping angle constant, if the curve of the edge is an asymptotic line of a hyperbolic part of the surface, then the edge behave as a *rigid* curve (theorem 8.2). This theorem can be considered as a generalization of the familiat rigidity of a straight edge.

We then consider surfaces with two edges, and obtain some new rigidity results concerning developable and hyperbolic surfaces. In particular we prove the geometrical rigidity of general hyperbolic surfaces with two keeping constant angle edges, which join together at one point M and satisfying a condition, we call PLASP†: we suppose there is no asymptotic line issued from M separating the edges (theorems 9.5, 9.6, 9.7), see figure 9. The proof of this last result is based on a uniqueness theorem of a non-classical variant of *Goursat problem for hyperbolic system*, given

†From the french : *Pas de Lignes Asymptotiques Séparant les Plis.*

in Appendix (theorem A.2).

In section 10, we recall some elements of asymptotic theory of thin elastic shells and the difficulties of their numerical study. In particular, we recall the notion of *sensitive problem*, for which we exhibit a new example involving three hyperbolic surfaces joined together in two keeping angle constant and satisfying the PLASP conditions.

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## 1.1 General notations

In this paper, Greek indices and exponents vary in the set {1,2}, Latin indices and exponents vary in the set {1,2,3}, and the repeated index or exponent convention for summation is used (Einstein's convention). The Euclidean inner product, the vector product and the Euclidean norm of vectors $\mathbf{u}, \mathbf{v} \in R^3$ are denoted $\mathbf{u}.\mathbf{v}$, $\mathbf{u}\wedge\mathbf{v}$ and $|\mathbf{u}|$. The vectors, vector fields, and vector spaces are denoted in bold characters: $L^2$ denotes the space of square integrable real functions, while $\mathbf{L}^2$ denotes the space of square integrable vector functions. The norm of a square integrable function $\mathbf{u}$ is denoted by $\|\mathbf{u}\|_{L^2}$. The partial derivatives are denoted in index preceded by a comma: $\frac{\partial \mathbf{u}}{\partial y^\alpha} = \mathbf{u}_{,\alpha}$. The classical Sobolev spaces are denoted by $H^m$ (or $\mathbf{H}^m$, if it is a vector space). The spaces of continuous functions with continuous $m$-order derivatives are denoted by $C^m$. The space of infinitely differentiable functions ($C^\infty$) with compact support is denoted by $\mathcal{D}$ and its dual, the Distribution space, is denoted by $\mathcal{D}'$. The space of the admissible displacements is denoted by $\mathbf{V}$. The space of the inextensional displacements is denoted by $\mathbf{G}$.

All along the paper, a surface will be defined by a map $(\Omega, \mathbf{r})$, where $\Omega$ is a domain of $R^2$, and $\mathbf{r}$ denotes the position vector.

# 2 Elements of theory of surfaces

The aim of this section is to recall some of the principal objects used in elementary theory of surfaces, which is essential for the rest of this paper. We refer to various manuals for a complete account of surface theory such as [18], [34] or [33].

Let E be the Euclidean space and let S be a surface in E, defined by the map $(\Omega, \mathbf{r})$ where $\Omega$ is a connected open set of $R^2$ and the position vector $\mathbf{r}$ is a $C^2$ mapping from $\Omega$ into E. The parameters will be denoted by $y^1$ and $y^2$, $\mathbf{r} : (y^1, y^2) \in \Omega \longmapsto \mathbf{r}(y^1, y^2) \in S$.

Figure 1: *A parameterized surface S.*

The curves defined by $\mathbf{r}(y^\alpha = \text{const.})$ are called coordinates curves of S and the parameters $(y^1, y^2)$ are called curvilinear coordinates. On each point of the surface, the *covariant basis* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is defined by

$$\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} = \frac{\partial \mathbf{r}}{\partial y^\alpha}, \ \mathbf{a}_3 = \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{\|\mathbf{a}_1 \wedge \mathbf{a}_2\|}. \tag{2.1}$$

We also define the coefficient

$$a = \|\mathbf{a}_1 \wedge \mathbf{a}_2\|^2. \tag{2.2}$$

As we suppose that the map $(\Omega, \mathbf{r})$ is sufficiently smooth, the tangent vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are uniquely defined. We also suppose they are never colinear, in order the triplet $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ to be a basis of $R^3$. As it is not orthonormal in general, neither orthogonal, it is associated with a dual basis $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$ called *contravariant basis* and defined by $\mathbf{a}^i.\mathbf{a}_j = \delta^i_j$, where $\delta^i_j$ denotes the Kronecker symbol.

## 2.1 First and second fundamental forms on a surface

Let $\mathbf{M}$ be a point of the surface § and $T_{\mathbf{M}}$§ the tangent plane to § on $\mathbf{M}$.

**Definition 2.1.** $\mathbf{I_M}$, *the first fundamental form of § on* $\mathbf{M}$ *is the quadratic form on* $T_{\mathbf{M}}$§*, the coefficients of which (of its matrix representation in the frame* $\mathbf{a}_1, \mathbf{a}_2$*) are given by*

$$a_{\alpha\beta} = \mathbf{a}_\alpha.\mathbf{a}_\beta = \mathbf{a}_\beta.\mathbf{a}_\alpha = a_{\beta\alpha}. \tag{2.3}$$

On each point of $S$, the first fundamental form is defined as the square of the differential form d$\mathbf{r}$. In this paper, it is denoted $\mathbf{I}$=d$\mathbf{r}$.d$\mathbf{r}$. It can also be seen as the restriction on the tangent plane of the Euclidean metric. It induces a metric on the surface. $\mathbf{I}$ is also sometimes denoted d$s^2$.

**Definition 2.2.** $\mathbf{II_M}$, *the second fundamental form of § on* $\mathbf{M}$ *is the quadratic form on* $T_{\mathbf{M}}$§*, the coefficients of which (of its matrix representation in the frame* $\mathbf{a}_1, \mathbf{a}_2$*) are given by*

$$b_{\alpha\beta} = \mathbf{a}_3.\mathbf{a}_{\alpha,\beta} = \mathbf{a}_3.\mathbf{a}_{\beta,\alpha} = b_{\beta\alpha}. \tag{2.4}$$

The two fundamental forms have geometrical interpretation. The former measures the intrinsic length of the surface whereas the later measures the curvatures of the surface.

The derivatives of the covariant (resp.contravariant) basis can be expressed in the covariant (resp. contravariant) basis by the Gauss and Weingarten formulas

$$\begin{array}{l} \mathbf{a}_{\alpha,\beta} = \Gamma^1_{\alpha\beta}\mathbf{a}_1 + \Gamma^2_{\alpha\beta}\mathbf{a}_2 + b_{\alpha,\beta}\mathbf{a}_3 \\ \mathbf{a}^\alpha_{,\beta} = -\Gamma^\alpha_{\beta 1}\mathbf{a}^1 - \Gamma^\alpha_{\beta 2}\mathbf{a}^2 - b^\alpha_\beta\mathbf{a}^3 \\ \mathbf{a}_{3,\alpha} = b^1_\alpha\mathbf{a}_1 + b^2_\alpha\mathbf{a}_2 = -b_{\alpha 1}\mathbf{a}^1 - b_{\alpha 2}\mathbf{a}^2, \end{array} \tag{2.5}$$

where the coefficients $\Gamma^\lambda_{\alpha\beta}$ are the *Christoffel's symbols*, given by

$$\Gamma^\lambda_{\alpha\beta} = \mathbf{a}^\lambda.\mathbf{a}_{\alpha,\beta}, \tag{2.6}$$

and where $b^\alpha_\beta = b_{\beta\lambda}a^{\lambda\alpha}$, with $a^{\lambda\alpha} = \mathbf{a}^\lambda.\mathbf{a}^\alpha$.

For any vector field $\mathbf{u}$ defined on the surface, with covariant components $u_i$ ($\mathbf{u} = u_1\mathbf{a}^1 + u_2\mathbf{a}^2 + u_3\mathbf{a}^3$,), by virtue of (2.5), we can express the partial derivative of $\mathbf{u}$ in the form

$$\begin{array}{ll} \mathbf{u}_{,\alpha} & = [u_{1,\alpha} - \Gamma^\lambda_{1\alpha}u_\lambda - b_{1\alpha}u_3]\mathbf{a}^1 \\ & +[u_{2,\alpha} - \Gamma^\lambda_{2\alpha}u_\lambda - b_{2\alpha}u_3]\mathbf{a}^2 \\ & +[u_{3,\alpha} - b^\lambda_\alpha u_\lambda]\mathbf{a}^3. \end{array} \tag{2.7}$$

## 2.2 Asymptotic directions and classification of surfaces

Consider now, on a point $\mathbf{M}$ of §, the function which associates to any unit vector X, tangent to § on $\mathbf{M}$, the real $\mathbf{II_M}$(X). This function reaches its two extrema, which we shall denote by $k_1$ and $k_2$, called *principal curvatures* of § on $\mathbf{M}$. The corresponding tangent directions X are called *principal directions.*

**Definition 2.3.** *A direction for which the function* $\mathbf{II_M}$ *vanishes is called an asymptotic direction.*

**Definition 2.4.** *An asymptotic line of a surface* $S$ *is a curve of* $S$*, the tangents of which are asymptotic directions.*

We can give a geometrical interpretation of an asymptotic line. Either they are lines the osculating plane[‡]of which coincide with the tangent plane, either they are straight lines.

As the second fundamental form $\mathbf{II}$ is a quadratic form, a direction X of $T_{\mathbf{M}}$§ is an asymptotic direction if and only if X= $(a\mathbf{a}_1 + b\mathbf{a}_2)$ is a root of the second order polynomial

$$\mathbf{II}(X) = a^2 b_{11} + 2ab b_{12} + b^2 b_{22}.$$

Let $\Gamma$ be a curve of the surface $S$, defined by

$$\Gamma = \{(y^1, y^2) \in \Omega / \Phi(y^1, y^2) = 0\}$$

A direction X $= (a, b)$ is tangent to the curve if and only if $a\phi_{,1} + b\phi_{,2} = 0$, therefore we have:

**Proposition 2.5.** $\Gamma$ *is an asymptotic line of* $S$ *if and only if*

$$(\phi_{,2})^2 b_{11} - 2\phi_{,1}\phi_{,2} b_{12} + (\phi_{,1})^2 b_{22} = 0. \tag{2.8}$$

*everywhere on* $\Gamma$*.*

We see that the existence of asymptotic directions (and therefore the existence of asymptotic lines) depends on the sign of the discriminant

$$\Delta = (b_{12})^2 - b_{11} b_{22} = -det(\mathbf{II_M}).$$

The notion of asymptotic direction induces the usual classification of a surface:

**Definition and proposition 2.6.** *Let* $\mathbf{M}$ *be a point of* $S$*, then*

*(i) if* $det(\mathbf{II_M}) > 0$*, there is no asymptotic direction at the point* $\mathbf{M}$*, which is said to be elliptic.*

*(ii) if* $det(\mathbf{II_M}) < 0$*, there are exactly two distinct asymptotic directions at the point* $\mathbf{M}$*, which is said to be hyperbolic.*

*(iii) if* $det(\mathbf{II_M} = 0)$*=0 and* $\mathbf{II_M} \neq 0$*, there is a unique asymptotic direction at the point* $\mathbf{M}$*, which is said to be parabolic.*

*(iv) if* $\mathbf{II_M}$*=0, the point* $\mathbf{M}$ *is said to be flat.*

Accordingly, a surface will be said uniformly elliptic, hyperbolic or parabolic if all the points of §are uniformly elliptic, hyperbolic or parabolic, respectively.

[‡] The osculating plane of a curve can be seen as the plane defined by three consecutive points, as the tangent can be seen as the straight line defined by two consecutive points.

**Remark 2.7.** The product $\mathrm{K} = k_1.k_2$ of the two principal curvatures is the Gaussian curvature of the surface on a point $\mathbf{M}$. We have the classical formula $\mathrm{K} = \frac{\det(\mathbf{II_M})}{\det(\mathbf{I_M})}$. As the Gaussian curvature has the same sign as $\det(\mathbf{II_M})$, hyperbolic surfaces (resp elliptic surfaces) are often designed as surfaces with negative (resp. positive) Gaussian curvature. □

This classification has, of course, a geometric interpretation.

Figure 2: *An elliptic point and a hyperbolic point.*

In the case of an elliptic point, the positivity of the Gaussian curvature K means that the principal curvatures are of the same sign; a neighborhood of $\mathbf{M}$ is entirely at one side of tangent plane.

On the contrary, if K is negative, the principal curvatures are of opposite sign and the surface go through the tangent plane where $\mathbf{II_M}$ vanishes, i.e on the asymptotic directions.

The parabolic surfaces are the *developable* surface, that is to say they are *ruled* surfaces such that the unit normal to the surface is constant along the *generators*.

A ruled surface is a surface such that it can be given by a map $(\Omega, \mathbf{r})$ of the form

$$\mathbf{r}(y^1, y^2) = \mathbf{c}(y^1) + y^2\mathbf{g}(y^1) \tag{2.9}$$

where $\mathbf{c}$ defines a smooth curve, called *directrix* of the ruled surface, and $\mathbf{g}$ is a vector field on $\mathbf{c}$. The straight lines at $y^1 = const.$ are the *generators*.

With a map in the form (2.9), various coefficients of the surface simplify:

$$b_{22} = \Gamma_{22}^{\alpha} = 0.$$

It is then easily seen that :

**Proposition 2.8.** *Let $S$ be a ruled surface given by (2.9), then $K \leq 0$ and:*
*(i) $S$ is developable if and only if $b_{12} = 0$, or equivalently $K = 0$.*
*(ii) The non-developable ruled surfaces are hyperbolic.*

## 2.3 Special coordinates on a surface

A very useful property of surfaces is the possibility of choosing a map, in order to get some simplification in the coefficients. This property is deduced from the classical reduction of a quadratic form (see [35] or [13]).

**Proposition 2.9.** *Let $S$ be a hyperbolic surface. There is a map, called asymptotic coordinate map, such that the coordinate curves coincide with the asymptotic lines. In this map the coefficients of second fundamental form simplify:*

$$b_{11} = b_{22} = 0 \text{ and } b_{12} \neq 0. \tag{2.10}$$

**Proposition 2.10.** *Let $S$ be an elliptic surface, there is a map, called isometric-conjugate coordinate map, such that the coefficients of the second fundamental form simplify:*

$$b_{11} = b_{22} = \sqrt{aK} \text{ and } b_{12} = 0. \tag{2.11}$$

More generally, we can choose, for any kind of surface, a map such that the coordinate curves are orthogonal, see [18]: lemma 3.2.2:

**Proposition 2.11.** *In any surface, there is a map called orthogonal coordinate map, such that we have:*

$$b_{12} = 0. \tag{2.12}$$

# 3 Inextensional Displacement on a Surface.

In this section, we give the classical definition of inextensional displacements of a surface. Inextensional displacements are essentially displacements that satisfy a P.D.E system, called *bending system*, the "characteristics" of which are the asymptotic lines of the surface. We emphasize on the notion of *infinitesimal rotation*, the derivatives of which will prove to be fundamental in the sequel.

Let $S$ be a surface defined by a map $(\Omega, \mathbf{r})$. The first fundamental form of $S$ is denoted by $\mathbf{I}$=d$\mathbf{r}$.d$\mathbf{r}$. The surface is deformed with a displacement $\mathbf{u}$, supposed to be small in order to linearize with respect to $\mathbf{u}$. The first fundamental form of the deformed surface is

$$\tilde{\mathbf{I}} = d(\mathbf{r} + \mathbf{u}).d(\mathbf{r} + \mathbf{u}).$$

**Definition 3.1.** *A displacement $\mathbf{u}$ is said inextensional (or infinitesimal bending), if it satisfies the bending system (also called rigidity system):*

$$d\mathbf{r}.d\mathbf{u} = 0, \tag{3.1}$$

*in other words if:*

$$\mathbf{u}_{,\alpha}.\mathbf{a}_\beta + \mathbf{u}_{,\beta}.\mathbf{a}_\alpha = 0. \tag{3.2}$$

Actually, the bending system (3.2) is a P.D.E. system of three equations with three unknowns (the three components of the displacement). Inextensional displacements are displacements that leave the first fundamental form unchanged in the linearized sense:

$$\tilde{\mathbf{I}} - \mathbf{I} = d(\mathbf{r} + \mathbf{u}).d(\mathbf{r} + \mathbf{u}) - d\mathbf{r}.d\mathbf{r} = 2d\mathbf{r}.d\mathbf{u} + d\mathbf{u}.d\mathbf{u}.$$

The linearized variation of the first fundamental form is twice the so called linearized tensor of deformation and denoted by $\boldsymbol{\gamma}$, usually expressed in covariant components:

$$\gamma_{\alpha\beta} = \frac{1}{2}[\mathbf{u}_{,\alpha}.\mathbf{a}_{\beta} + \mathbf{u}_{,\beta}.\mathbf{a}_{,\alpha}] = \frac{1}{2}(u_{\beta,\alpha} + u_{\alpha,\beta}) - \Gamma^{\lambda}_{\alpha\beta}u_{\lambda} - b_{\alpha\beta}u_3. \tag{3.3}$$

Let $\mathbf{V}$ be the space of admissible displacements, choosen to fit with the theory of linear thin elastic shells, see section 10:

$$\mathbf{V} = \{\mathbf{v} \in H^1 \times H^1 \times H^2 + \text{ Kinematic boundary conditions}\}.$$

**Definition 3.2.** $\mathbf{G} = \{\mathbf{v} \in \mathbf{V}/d\mathbf{r}.d\mathbf{u} = 0\} = \{$*admissible inextensional displacements*$\}$.

**Definition 3.3.** *A displacement* $\mathbf{u}$ *is said rigid (or to be a trivial bending) if there are two constant vectors* $C_1$ *and* $C_2$*, such that :*

$$\mathbf{u} = C_1 \wedge \mathbf{r} + C_2. \tag{3.4}$$

Rigid displacements are trivial solutions of the bending system. Furthermore they are the only displacements that leave invariant both first and second fundamental form, see the rigid movement lemma of Bernadou and Ciarlet [4].

**Definition 3.4.** *Let* $S$ *be a surface (resp. a portion of surface).* $S$ *is said geometrically rigid or inhibited if* $\mathbf{G}$ *on* $S$ *is included in the set of rigid displacements.*

## 3.1 The bending system.

The bending system (3.2) can be expressed in local coordinates. For a displacement $\mathbf{u}$, where $(u_1, u_2, u_3)$ denote its covariant components, developing with (2.7), the bending system (3.2) can be rewritten as:

$$\left\{\begin{array}{l} u_{1,1} = \Gamma^{\lambda}_{11}u_{\lambda} + b_{11}u_3 \\ u_{2,2} = \Gamma^{\lambda}_{22}u_{\lambda} + b_{22}u_3 \\ u_{1,2} + u_{2,1} = 2\Gamma^{\lambda}_{12}u_{\lambda} + 2b_{12}u_3, \end{array}\right. \tag{3.5}$$

The linear first order P.D.E. system (3.5) is called *covariant coordinates bending system.*

Classically, in the case of a plane surface, since all the coefficients of the surface vanishes, one can see from the bending system, that the inextensional displacements on a plane are the normal displacements (modulo rigid displacements).

We suppose now that $S$ is not a plane neither it contains any flat points. Let us choose an orthogonal coordinate map (see proposition 2.11). In such a map, we have $b_{12} = 0$. It is then possible to "reduce" the system (3.5) by "eliminating" the normal component $u_3$. Combining the first and second equations of (3.5) we obtain:

$$\left\{\begin{array}{l} u_{1,2} + u_{2,1} = 2\Gamma^{\lambda}_{12}u_{\lambda} \\ b_{22}u_{1,1} - b_{11}u_{2,2} = (b_{22}\Gamma^{\lambda}_{11} - b_{11}\Gamma^{\lambda}_{22})u_{\lambda}. \end{array}\right. \tag{3.6}$$

The system (3.6) is called *reduced bending system*. We can see that, if the tangential components $u_1$ and $u_2$ satisfying (3.6) are given, then the first or second equation of (3.5) determine the normal component $u_3$ in a unique way. The reduced bending system (3.6) has the fundamental property:

**Proposition 3.5.** *The characteristics of the reduced bending system (3.6) coincide with the asymptotic lines of S. In other words, the first order P.D.E. reduced bending system (3.6) is elliptic or hyperbolic, whether the surface S is elliptic or hyperbolic.*

**Proof.** Let $\Gamma = \{(y^1, y^2)/\Phi(y^1, y^2) = 0\}$ be a curve of $\Omega$ (and therefore determining a curve on S), where $\Phi$ is a sufficiently smooth function. Then, $\Gamma$ is a characteristic of the bending system (3.6) if $\Phi$ satisfies the equation:

$$\det \left[ \Phi_{,1} \begin{pmatrix} 0 & 1 \\ b_{22} & 0 \end{pmatrix} + \Phi_{,2} \begin{pmatrix} 1 & 0 \\ 0 & -b_{11} \end{pmatrix} \right] = 0. \qquad (3.7)$$

In other words if

$$(\Phi_{,1})^2 b_{22} + (\Phi_{,2})^2 b_{11} = 0. \qquad (3.8)$$

We recognize, in (3.8), the equation (2.8) (with $b_{12} = 0$) which defines the asymptotic lines of the surface.□

**Remark 3.6.** For any surfaces given by a Cartesian map $(\mathbf{r}, \Omega)$, in a fixed orthonormal frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$:

$$\mathbf{r}(y^1, y^2) = y^1 \mathbf{e}_1 + y^2 \mathbf{e}_2 + \phi(y^1, y^2)\mathbf{e}_3,$$

if the $u_1, u_2, u_3$ denote the Cartesian coordinates of a displacement $\mathbf{u}$, the bending system can be rewritten in Cartesian components :

$$\left\{ \begin{array}{rcl} u_{1,1} + \phi_{,1} u_{3,1} & = & 0 \\ u_{2,2} + \phi_{,2} u_{3,2} & = & 0 \\ u_{1,2} + u_{2,1} + \phi_{,1} u_{3,2} + \phi_{,2} u_{3,1} & = & 0. \end{array} \right. \qquad (3.9)$$

The system (3.9) has the particularity to be *non-Kowaleskian* (i.e. all curves are characteristics). But it is "equivalent", in a certain sense, to a second order partial differential equation satisfied by the *vertical* component $u_3$ (see G. Darboux [14]):

$$\phi_{,22} u_{3,11} - 2\phi_{,12} u_{3,12} + \phi_{,11} u_{3,22} = 0, \qquad (3.10)$$

the characteristics of which coincides with the asymptotic lines of the surface. □

Depending on the geometric nature of the considered surface, without any loss of generality, one can take advantage of choosing adequate map in order to simplify the bending system.

In the case of a hyperbolic surface, one can choose an asymptotic coordinate map (see proposition 2.9), in which the coefficients of the second fundamental form simplify:

$$b_{11} = b_{22} = 0 \text{ and } b_{12} \neq 0.$$

The bending system (3.5) then reduces to:

$$\begin{cases} u_{1,1} = \Gamma_{11}^{\lambda} u_{\lambda} \\ u_{2,2} = \Gamma_{22}^{\lambda} u_{\lambda} \\ u_{1,2} + u_{2,1} = 2\Gamma_{12}^{\lambda} u_{\lambda} + 2b_{12} u_3. \end{cases} \tag{3.11}$$

One can see that the two first equations of (3.11) form a hyperbolic first order partial differential system of two equations, under its so-called *diagonal* form. The third equation of (3.11), determines the normal component of the displacement if the tangential components are given.

In the case of an elliptic surface, one can choose an isometric conjugate coordinate map (see proposition 2.10), in which the coefficients of the second fundamental form simplify:

$$b_{11} = b_{22} \neq 0 \text{ and } b_{12} = 0.$$

The reduced bending system (3.6) becomes:

$$\begin{cases} u_{1,2} + u_{2,1} = 2\Gamma_{12}^{\lambda} u_{\lambda} \\ u_{1,1} - u_{2,2} = (\Gamma_{11}^{\lambda} - \Gamma_{22}^{\lambda}) u_{\lambda}. \end{cases} \tag{3.12}$$

Let then, $w = u_1 + iu_2$ and $z = y^1 + iy^2$, where $i^2 = -1$, we have:

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial y^1} - i\frac{\partial w}{\partial y^2}\right) \text{ and } \frac{\partial w}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial y^1} + i\frac{\partial w}{\partial y^2}\right). \tag{3.13}$$

The system (3.12) is then equivalent to the differential equation:

$$\frac{\partial w}{\partial z} = Aw + B\overline{w}. \tag{3.14}$$

Where

$$\begin{aligned} A &= \tfrac{1}{2}[\Gamma_{11}^1 - \Gamma_{22}^1 + 2\Gamma_{12}^2 + 2i\Gamma_{12}^1 - i\Gamma_{11}^2 + i\Gamma_{22}^2] \\ B &= \tfrac{1}{4}[\Gamma_{11}^1 - \Gamma_{22}^1 - 2\Gamma_{12}^2 + 2i\Gamma_{12}^1 + i\Gamma_{11}^2 - i\Gamma_{22}^2]. \end{aligned}$$

By *definition*, a solution of the differential equation (3.14) is a *generalized analytic function*[§]which has many properties similar to most of analytic functions properties (maximum principle, generalized Cauchy formulas, etc..). It is the theory developed in the monograph of I.N. Vekua [35], used to obtain various rigidity results for elliptic surfaces with a keeping constant angle edge. We shall recall one of them in proposition 5.1.

We also indicate, that the bending system (3.6), in the case of elliptic surface, is *elliptic in Douglis-Nirenberg sense.* This aspect have been developed in [15] where some rigidity results are proved, similar to those found in Vekua's monograph.

[§]Generalized analytic functions are also called *pseudo-analytic functions*, see [35] or [5] or even [13].

We end this subsection with a regularity property of inextensional displacements, see also [15]:

**Proposition 3.7.** $\mathbf{G} \subset \mathbf{H}^2$.

**Proof.** It is easily seen from (3.5), which gives $u_{1,1} \in H^1$ $u_{2,2} \in H^1$ and $(u_{1,2}+u_{2,1}) \in H^1$. Then $u_{1,11}, u_{1,12}, u_{2,21},\ u_{2,22}, (u_{1,21}+u_{2,11})$ and $(u_{1,22}+u_{2,12)}) \in L^2$. Consequently, we have moreover $u_{1,22}$ and $u_{2,11} \in L^2$. □

## 3.2 Infinitesimal rotation field

Let §be a surface with a map $(\Omega, \mathbf{r})$ and $\mathbf{u} \in \mathbf{G}$ be an inextensional displacement. Classically, see [14], [33] or [35], $\mathbf{u}$ determines a unique vector field $\boldsymbol{\omega}$ satisfying

$$d\mathbf{u} = \boldsymbol{\omega} \wedge d\mathbf{r},$$

or equivalently

$$\begin{cases} \mathbf{u}_{,1} = \boldsymbol{\omega} \wedge \mathbf{a}_1 \\ \mathbf{u}_{,2} = \boldsymbol{\omega} \wedge \mathbf{a}_2. \end{cases} \tag{3.15}$$

More precisely, we have:

**Proposition 3.8.** *Let* $\mathbf{u}$ *be an inextensional displacement, there is a unique vector field, called infinitesimal rotation vector field*[¶]*associated to* $\mathbf{u}$*, satisfying the equations (3.15). Moreover, the rotation field satisfies necessarily*

$$\boldsymbol{\omega}_{,1} \wedge \mathbf{a}_2 = \boldsymbol{\omega}_{,2} \wedge \mathbf{a}_1. \tag{3.16}$$

*We have the explicit expression, where the* $w^i$ *denote the contravariant components of the rotation field* $\boldsymbol{\omega}$*:*

$$\begin{cases} w^1 = \frac{1}{a}(u_{3,2} - b_2^\lambda u\lambda) \\ w^2 = \frac{1}{a}(u_{3,1} - b_1^\lambda u\lambda) \\ w^3 = \frac{1}{a}\left[\frac{1}{2}(u_{2,1} - u_{1,2}) - \Gamma_{12}^\lambda u_\lambda - b_{12}u_3\right]. \end{cases} \tag{3.17}$$

**Proof.** Let us first show the uniqueness of the associated rotation field. Let $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ be two vector fields satisfying to (3.15) for a same inextensional displacement $\mathbf{u}$. (3.15) then implies

$$(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \wedge \mathbf{a}_1 = (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \wedge \mathbf{a}_2 = 0,$$

and thereby, we obtain $\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2 = 0$.

Let us write now, the Schwartz equality $\mathbf{u}_{,12} = \mathbf{u}_{,21}$. With (3.15), it gives immediately the equality (3.16).

The explicit expression (3.17) is then obtained by developing, in (3.15), the partial derivatives $\mathbf{u}_{,\alpha}$ in covariant components and the rotation field $\boldsymbol{\omega}$ in contravariant components. It suffices then to identify each terms in the contravariant basis. □

[¶]We shall say *rotation field*, as there is no ambiguity.

**Remark 3.9.** A vector field given by formula (3.17) appears in the expression of the variation of the second fundamental form (see [4]). But, it is a rotation field in the sense of (3.16) if and only if the corresponding displacement is inextensional. □

**Remark 3.10.** It follows from (3.17) and proposition 3.7 that a rotation field belongs to the Sobolev's space $\mathbf{H}^1$. It gives easily the following estimates: there is a positive constant $C$ such that

$$\forall \mathbf{u} \in \mathbf{G}, \quad \begin{array}{l} \|\boldsymbol{\omega}\|_{\mathbf{L}^2(\Omega)} \leq C\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \\ \|\boldsymbol{\omega}_{,\alpha}\|_{\mathbf{L}^2(\Omega)} \leq C\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}. \ \square \end{array} \tag{3.18}$$

**Definition 3.11.** *The space* $\mathbf{W}$ *of rotation vector field on* § *is*

$$\mathbf{W} = \{\boldsymbol{\omega} \in H^1 / \boldsymbol{\omega}_{,1} \wedge \mathbf{a}_2 = \boldsymbol{\omega}_{,2} \wedge \mathbf{a}_1\}.$$

This last definition is justified by a reciprocal to the proposition 3.8:

**Proposition 3.12.** *Let* $\boldsymbol{\omega} \in \mathbf{W}$ *be a rotation field on* §*,* $\boldsymbol{\omega}$ *determines an inextensional displacement* $\mathbf{u}$*, which is unique modulo a rigid displacement. We have the explicit expression, modulo a rigid displacement*

$$\begin{array}{rl} \mathbf{u}(y^1, y^2) = & \int_0^{y^1} \boldsymbol{\omega}_{,1}(t, y^2) \wedge [\mathbf{r}(y^1, y^2) - \mathbf{r}(t, y^2)]dt \\ & + \int_0^{y^2} \boldsymbol{\omega}_{,2}(0, s) \wedge [\mathbf{r}(y^1, y^2) - \mathbf{r}(0, s)]ds. \end{array} \tag{3.19}$$

**Proof.** Let us first remark that a rotation field belongs, a priori, to the Sobolev space $\mathbf{H}^1$. Therefore, the trace of its first order partial derivatives does not make sense *a priori.* In fact, the expression (3.19) must be considered as the continuous extension of the integral operator from $\mathbf{L}^2$ to $\mathbf{L}^2$. The notation of (3.19) is then abusive, but not ambiguous.

Let $\boldsymbol{\omega}$ be a rotation field and let $\mathbf{u}$ be a displacement satisfying the two equations (3.15). Making the scalar product of the two equalities of (3.15) respectively with $\mathbf{a}_1$ and $\mathbf{a}_2$, we immediately see that the bending system is satisfied by $\mathbf{u}$. Thus, $\mathbf{u}$ is inextensional.

Formally, the expression (3.19) is obtained by quadratures for smooth rotation fields. It is then justified for all rotation fields in $\mathbf{H}^1$, as we said, by continuous extension of the integral operator from $\mathbf{L}^2$ to $\mathbf{L}^2$. □

**Remark 3.13.** The expression (3.19) is not unique (although, the inextensional displacement is uniquely determined modulo a rigid displacement). One can choose different paths and take, for instance:

$$\begin{array}{rl} \mathbf{u}(y^1, y^2) = & \int_0^{y^1} \boldsymbol{\omega}_{,1}(t, 0) \wedge [\mathbf{r}(y^1, y^2) - \mathbf{r}(t, 0)]dt \\ & + \int_0^{y^2} \boldsymbol{\omega}_{,2}(y^1, s) \wedge [\mathbf{r}(y^1, y^2) - \mathbf{r}(y^1, s)]ds. \end{array} \tag{3.20}$$

The independence of the "quadrature" is insured by the compatibility equation (3.16). It is even possible to combine (3.19) and (3.20) to obtain valid explicit expression for complex form of $\Omega$ (non convex for instance, or with a hole). □

The last proposition induces some criteria of rigidity:

**Proposition 3.14.** *Let § be a surface and let* $\mathbf{u}$ *be an inextensional displacement on* $S$*,* $\mathbf{u}$ *is a rigid displacement if and only if the associated rotation vector field* $\boldsymbol{\omega}$ *is constant on* $S$*.*

**Proof.** By definition, if $\mathbf{u}$ is rigid then $\boldsymbol{\omega}$ is constant. Conversely, if the rotation field $\boldsymbol{\omega}$ is constant, the expression (3.19) immediately shows that the displacement $\mathbf{u}$ is rigid. □

The last rigidity criteria can be "restricted" along a curve :

**Proposition 3.15.** *Let* $\Gamma$ *be a curve of §. Let* $\mathbf{u}$ *be an inextensional displacement on* $S$ *and let* $\boldsymbol{\omega}$ *be its associated rotation field. If* $\boldsymbol{\omega}$ *is constant along the curve* $\Gamma$*, then* $\mathbf{u}$ *is rigid along* $\Gamma$*.*

**Proof.** Without any loss of generality, we can choose a map $(\Omega, \mathbf{r})$ of § such that the curve $\Gamma$ is defined by the coordinate curve $y^2 = 0$. Integrating the first equation of (3.15) along $y^2 = 0$ gives:

$$\mathbf{u}(y^1, 0) = \boldsymbol{\omega} \wedge [\mathbf{r}(y^1, 0) - \mathbf{r}(0, 0)]. \tag{3.21}$$

In other words, as $\boldsymbol{\omega}$ is constant along the curve $y^2 = 0$, (3.21) shows that $\mathbf{u}$ is rigid on $y^2 = 0$, i.e. on $\Gamma$. □

**Remark 3.16.** The rigidity criteria along a curve of proposition 3.15 is sufficient but not necessary in general. The expression (3.21) implies a rigid displacement on the curve if and only if the component of $\boldsymbol{\omega}$, transverse to $\Gamma$, is constant. The tangential component can be non-constant. The case of developable surface gives an example: The generatrices behave as rigid lines but the corresponding rotation field is not constant along them, see the proposition 4.9. □

# 4 The Space $\mathbf{R}(S)$, Derivatives of Rotation Field.

We develop in this section the framework of a new approach of the theory of inextensional displacements. It consists on the introduction of a non-classical space, denoted $\mathbf{R}(S)$. It is the key of the proof of various new results of rigidity concerning hyperbolic and developable surfaces that we shall show in next sections.

In the expression (3.19), the first partial derivatives of a rotation vector field determine uniquely (modulo a rigid displacement) an inextensional displacement. Furthermore, one can express the rigidity criteria of the propositions 3.14 and 3.15 as a condition on the derivatives of a rotation field. These objects, first partial derivatives of rotation field, constitutes the elements of the space $\mathbf{R}$(§). They are non-classical objects, although similar to the bending field of I.N. Vekua [35].

The fundamental property of derivatives of rotation field is the following tangential property:

**Proposition 4.1.** *Let $\boldsymbol{\omega} \in \mathbf{W}$ be a rotation field on surface $S$. Then the normal component of first partial derivatives of $\boldsymbol{\omega}$ vanishes. In other words, the differential of $\boldsymbol{\omega}$ is tangent to §.*

**Proof.** It suffices to take the scalar product of the compatibility equation (3.16) with $\mathbf{a}_1$ (resp. $\mathbf{a}_2$) to obtain: $\boldsymbol{\omega}_{,1} \wedge \mathbf{a}_2.\mathbf{a}_1 = \boldsymbol{\omega}_{,2} \wedge \mathbf{a}_1.\mathbf{a}_1 = 0$
$\Longrightarrow \boldsymbol{\omega}_{,1}.\mathbf{a}_1 \wedge \mathbf{a}_2 = 0,\ (resp.)\ \boldsymbol{\omega}_{,2}.\mathbf{a}_1 \wedge \mathbf{a}_2 = 0.$ □

For any $\boldsymbol{\omega} \in \mathbf{W}$, let us introduce the new notation:

$$\begin{cases} \mathbf{w}_1 = \boldsymbol{\omega}_{,1} \\ \mathbf{w}_2 = \boldsymbol{\omega}_{,2}. \end{cases} \tag{4.1}$$

The equality (3.16) becomes:

$$\mathbf{w}_1 \wedge \mathbf{a}_2 = \mathbf{w}_2 \wedge \mathbf{a}_1, \tag{4.2}$$

and the vector fields $\mathbf{w}_1$ and $\mathbf{w}_2$ must satisfy the integrability condition:

$$\mathbf{w}_{1,2} = \mathbf{w}_{2,1}. \tag{4.3}$$

**Definition 4.2.**

$$\mathbf{R}(S) = \{(\mathbf{w}_1, \mathbf{w}_2) \in [\mathbf{L}^2(\Omega)]^2 \ / \ \mathbf{w}_1 \wedge \mathbf{a}_2 = \mathbf{w}_2 \wedge \mathbf{a}_1 \ \textit{and}\ \mathbf{w}_{1,2} = \mathbf{w}_{2,1}\}.$$

According to proposition 3.8, an inextensional displacement determines a unique rotation field and therefore, a unique couple $(\mathbf{w}_1, \mathbf{w}_2)$ of $\mathbf{R}(\S)$. Let us denote by R the linear mapping:

$$\begin{array}{ccc} \mathbf{G} & \stackrel{\mathrm{R}}{\longrightarrow} & \mathbf{R}(\S) \\ \mathbf{u} & \longmapsto & \mathrm{R}(\mathbf{u}) = (\mathbf{w}_1, \mathbf{w}_2). \end{array}$$

According to proposition 3.14, an inextensional displacement is rigid if and only if its associated rotation field is constant on $S$, i.e. if and only if R($\mathbf{u}$)=(0,0). In other words, we have:

**Proposition 4.3.** *The vector space $\mathbf{R}(\S)$ is isomorphic with the quotient space $\mathbf{G}/\{$rigid displacement$\}$. Consequently, the surface § is geometrically rigid if and only if $\mathbf{R}(S)$ is reduced to $\{(0,0)\}$.*

**Remark 4.4.** It is easy to see that the space $\mathbf{R}(\S)$ is closed in $[\mathbf{L}^2(\Omega)]^2$. Furthermore, according to remark 3.10, there is a positive constant $C$ such that

$$\|\mathrm{R}(\mathbf{u})\|_{[\mathbf{L}^2(\Omega)]^2} = \|(\mathbf{w}_1, \mathbf{w}_2)\|_{[\mathbf{L}^2(\Omega)]^2} \leq C\|\mathbf{u}\|_{\mathbf{H}^2}\ \forall \mathbf{u} \in \mathbf{G}. \tag{4.4}$$

Therefore, applying a Banach's classical theorem, we see that R is bicontinuous and there is a positive constant $c$ such that

$$\inf_{\mathbf{v} \in \{rigid\ displ.\}} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^2} \leq c\|(\mathbf{w}_1, \mathbf{w}_2)\|_{[\mathbf{L}^2(\Omega)]^2}\ \forall \mathbf{u} \in \mathbf{G}.\ \square \tag{4.5}$$

## 4.1 The derived bending system

According to the tangential property of proposition 4.1, the normal component of elements of $\mathbf{R}(\S)$ vanishes. This permits us to have relatively simple expressions in contravariant components. If we denote $\mathbf{w}_\alpha = w_\alpha^\lambda \mathbf{a}_\lambda$, developing (4.3) in covariant components, gives

$$\left[w_{1,2}^\gamma + \Gamma_{2\lambda}^\gamma w_1^\lambda\right] \mathbf{a}_\gamma + \left[b_{2\lambda} w_1^\lambda\right] \mathbf{a}_3 = \left[w_{2,1}^\gamma + \Gamma_{1\lambda}^\gamma w_2^\lambda\right] \mathbf{a}_\gamma + \left[b_{1\lambda} w_2^\lambda\right] \mathbf{a}_3.$$

Identifying then, on each term of the covariant basis, we obtain a first order P.D.E system, we call *derived bending system*:

$$\left\{\begin{array}{lcl} w_{1,2}^1 + \Gamma_{2\lambda}^1 w_1^\lambda & = & w_{2,1}^1 + \Gamma_{1\lambda}^1 w_2^\lambda \\ w_{1,2}^2 + \Gamma_{2\lambda}^2 w_1^\lambda & = & w_{2,1}^2 + \Gamma_{1\lambda}^2 w_2^\lambda \\ b_{12} w_1^1 + b_{22} w_1^2 & = & b_{11} w_2^1 + b_{12} w_2^2. \end{array}\right. \tag{4.6}$$

On another hand, developing in (4.2) gives

$$w_1^\lambda \mathbf{a}_\lambda \wedge \mathbf{a}_2 = w_2^\lambda \mathbf{a}_\lambda \wedge \mathbf{a}_1 \implies w_1^1[\mathbf{a}_1 \wedge \mathbf{a}_2] = w_2^2[\mathbf{a}_2 \wedge \mathbf{a}_1],$$

and therefore we obtain

$$w_1^1 + w_2^2 = 0. \tag{4.7}$$

With the relation (4.7), the derived bending system (4.6) is reduced to a P.D.E. of three equations with three unknowns. But, no derivatives occurs in the third equation of (4.6); it is a compatibility equation. Therefore, the system (4.6)-(4.7) can be reduced to a first order P.D.E system of two equations with two unknowns. The third equation of (4.6) with (4.7) can be rewritten as

$$b_{22} w_1^2 = b_{11} w_2^1 + 2 b_{12} w_2^2,$$

thus, the first two equations of (4.6) give the *reduced derived bending system:*

$$\begin{array}{rcl} w_{2,1}^1 + w_{2,2}^2 & = & f_1(w_2^1, w_2^2) \\ b_{22} w_{2,1}^2 - b_{11} w_{2,2}^1 - 2 b_{12} w_{2,2}^2 & = & f_2(w_2^1, w_2^2), \end{array} \tag{4.8}$$

where $f_1$ and $f_2$ are two functions depending on the coefficients of the surface and $w_2^1$ and $w_2^2$ (in an affine way); we shall not explicit them. The important point is the nature of the P.D.E. system (4.8).

A curve $\Gamma = \{(y^1, y^2)/\Phi(y^1, y^2) = 0\}$ is a characteristic curve of (4.8) if and only if

$$det \left[\Phi_{,1} \begin{pmatrix} 1 & 0 \\ 0 & b_{22} \end{pmatrix} + \Phi_{,2} \begin{pmatrix} 0 & 1 \\ -b_{11} & -b_{12} \end{pmatrix}\right] = 0,$$

that is to say, if

$$b_{22}[\Phi_{,1}]^2 - 2 b_{12} \Phi_{,1} \Phi_{,2} + b_{11} [\Phi_{,2}]^2 = 0.$$

In last equation, we recognize the expression of equation (2.8). We have proved:

**Proposition 4.5.** *The contravariant components of an element* $(\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{R}(\S)$ *satisfy the derived bending system (4.6)-(4.7), equivalent to the reduced bending system (4.8), the characteristics of which coincide with the asymptotic lines of* §.

Therefore, the reduced derived bending system is hyperbolic (resp. elliptic) if the surface is uniformly hyperbolic (resp. elliptic). This is naturally consistent with the proposition 3.5.

## 4.2 Trace on a curve of elements of R(§)

The elements of $\mathbf{R}(S)$ belongs to the space $\mathbf{L}^2$, but although they are not defined a priori, we shall see, in the sequel, that it is possible, in most of cases, to give a sense to their trace on a curve.

**Proposition 4.6.** *Let* $S$ *be a hyperbolic surface and* $(\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{R}(S)$. *If* $\Gamma$ *is a smooth curve, transversal to the asymptotic lines of* $S$, *then the trace of* $(\mathbf{w}_1, \mathbf{w}_2)$ *on* $\Gamma$ *make sense in the space* $[\mathbf{L}^2(\Gamma)]^2$.

**Proof.** Let us choose an asymptotic coordinate map for §. With the simplification $b_{11} = b_{22} = 0$ (see proposition 2.9), the third equation of (4.6) becomes

$$b_{12} w_1^1 = b_{12} w_2^2.$$

Combining with (4.7), we obtain

$$w_1^1 = w_2^2 = 0.$$

Consequently, the first two equations of (4.6) gives

$$\begin{cases} w_{2,1}^1 \in L^2(\Omega) \\ w_{1,2}^2 \in L^2(\Omega). \end{cases}$$

Therefore, according to classical trace theorem (see lemma A.1 in appendix), the trace of $w_2^1$ (resp. $w_1^2$) make sense on smooth curves transversal to the curves (asymptotic lines) defined at $y^1 = const.$ (resp. the asymptotic lines defined at $y^2 = const.$). $\square$

In the last proof, one can notice that if the curve $\Gamma$ is an asymptotic line, say defined at $y^2 = 0$, then the trace of $w_1^2$ still make sense on $\Gamma$. In other words, we have:

**Corollary 4.7.** *Let* $\boldsymbol{\omega}$ *be a rotation field on a hyperbolic surface* $S$. *If a smooth curve* $\Gamma$ *is transversal to the asymptotic lines of* $S$ *or if* $\Gamma$ *is itself an asymptotic line of* $S$, *then the derivative of* $\boldsymbol{\omega}$ *along the curve* $\Gamma$ *make sense in* $[\mathbf{L}^2(\Gamma)]^2$.

In the case of a elliptic surface, we choose an isometric conjugate coordinate map (see proposition 2.10), with the simplifications $b_{11} = b_{22}$ and $b_{12} \neq 0$. The derived bending system (4.6)-(4.7) then reduces to:

$$\left\{ \begin{array}{l} w^1_{1,2} - w^1_{2,1} = [\Gamma^1_{11} - \Gamma^1_{22}] - w^1_2 - [2\Gamma^1_{12}]w^1_1 \\ w^1_{1,1} + w^1_{2,2} = [\Gamma^2_{11} - \Gamma^2_{22}] - w^1_2 - [2\Gamma^2_{12}]w^1_1 \end{array} \right. \tag{4.9}$$

and

$$\left\{ \begin{array}{l} w^1_1 + w^2_2 = 0 \\ w^2_1 - w^1_2 = 0. \end{array} \right. \tag{4.10}$$

Deriving the first equation of (4.9) with respect to $y^1$ and adding it to the second equation of (4.9) which we derive with respect to $y^2$, we obtain

$$\Delta w^1_1 \in H^{-1}(\Omega).$$

And in analogous way, we have

$$\Delta w^2_2 \in H^{-1}(\Omega).$$

Here "$\Delta$" denotes the usual Laplace differential operator. Thus, according to classical *local* regularity theorem for elliptic differential operator (see [25]), if the surface is smooth enough (say $C^3$), then $w^1_1$ and $w^2_2$ (and therefore $\mathbf{w}_1$ and $\mathbf{w}_2$) have the regularity $H^1$ in any open subset strictly contained in $\Omega$. Consequently, the trace of $\mathbf{w}_1$ and $\mathbf{w}_2$ on a curve $\Gamma$ make sense in $\mathbf{L}^2$ if $\Gamma$ is interior.

In the interesting case of a trace on the boundary of $\Omega$, a theorem of trace on the boundary for solution of elliptic differential operator (see [23], theorem 5.1, p. 168-171), permits to the trace to make sense in $H^{-\frac{1}{2}}(\Gamma)$.

**Proposition 4.8.** *Let $S$ be a uniformly elliptic surface of class $C^3$, and let $\Gamma$ be a curve of $S$, possibly on the boundary of $S$. Then the trace on $\Gamma$ of $(\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{R}(S)$ make sense in in $[\mathbf{H}^{-\frac{1}{2}}(\Gamma)]^2$.*

The case of a developable surface is considered in the sequel, where we also derive the classical general explicit expression of inextensional displacement given with a complete proof for inextensional displacements of $\mathbf{G}$, using the framework of $\mathbf{R}(S)$.

Let § be a developable surface, given by a map $(\Omega, \mathbf{r})$, in the form (2.9):

$$\mathbf{r}(y^1, y^2) = \mathbf{c}(y^1) + y^2\mathbf{g}(y^1).$$

For such a developable surface, we have the simplifications

$$b_{22} = b_{12} = 0,\ \Gamma^\alpha_{22} = 0 \text{ and } b_{11} \neq 0. \tag{4.11}$$

The derived bending system (4.6)-(4.7) then reduces to

$$\left\{ \begin{array}{l} w^1_2 = 0 \\ w^1_{1,2} = -2\Gamma^1_{12}w^1_1 \\ w^2_{1,2} = -w^1_{1,1} - 2\Gamma^2_{12}w^1_1 \end{array} \right. \tag{4.12}$$

$$w_1^1 + w_2^2 = 0. \tag{4.13}$$

We see that the second and third equations of (4.12) are ordinary differential equations. They are easily integrated and give

$$\begin{cases} w_1^1 = \rho_1(y^1)E(y^1, y^2) = -w_2^2 \\ w_1^2 = \rho_2(y^1) + F(y^1, y^2) \\ w_2^1 = 0, \end{cases} \tag{4.14}$$

where the functions $E(y^1, y^2)$ and $F(y^1, y^2)$ are given by

$$E(y^1, y^2) = \exp[-\int_0^{y^2} 2\Gamma_{12}^1(y^1, t)dt] \text{ and } F(y^1, y^2) = -\int_0^{y^2} [w_{1,1}^1 - 2\Gamma_{12}^2 w_1^1](y^1, t)dt.$$

The functions $\rho_1$ and $\rho_2$ are *a priori* arbitrary in $L^2$, but in order the component $w_1^2$ to be in $L^2$, $\rho_1$ is necessarily in $H^1$. The corresponding fields $(\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{R}(\S)$ are

$$\begin{cases} \mathbf{w}_1 = \rho_1(y^1)E(y^1, y^2)\mathbf{a}_1 + [F(y^1, y^2) + \rho_2(y^1)]\mathbf{a}_2 \\ \mathbf{w}_2 = -\rho_1(y^1)E(y^1, y^2)\mathbf{a}_2. \end{cases} \tag{4.15}$$

Applying the proposition 3.12, we have proved:

**Proposition 4.9.** *Let § be a developable surface given by a map in the form (2.9). For any inextensional displacement* $\mathbf{u} \in \mathbf{G}$ *on §, there are two functions* $\rho_1 \in H^1$ *and* $\rho_2 \in L^2$*, such that, modulo a rigid displacement, we have*

$$\mathbf{u}(y^1, y^2) = \int_0^{y^1} [\rho_1(t)\mathbf{c}'(t) + \rho_2(t)\mathbf{g}(t)] \wedge [\mathbf{c}(y^1) - \mathbf{c}(t) + y^2\mathbf{g}(y^1)]dt. \tag{4.16}$$

In the special case of a cone, we have furthermore:

**Proposition 4.10.** *Let* $S$ *be a cone containing its summit* $\mathbf{X}$ *given by the map* $(\Omega, \mathbf{r})$ *with*

$$\mathbf{r}(y^1, y^2) = \mathbf{X} + y^2\mathbf{g}(y^1),$$

*such that* $\mathbf{g}$ *and* $\mathbf{g}'$ *are linearly independent. For any inextensional displacement* $\mathbf{u} \in \mathbf{G}$ *on §, there is a function* $\rho \in L^2$ *such that, modulo a rigid displacement, we have*

$$\mathbf{u}(y^1, y^2) = \left[\int_0^{y^1} [\rho(t)\mathbf{g}(t)dt\right] \wedge y^2\mathbf{g}(y^1). \ \square \tag{4.17}$$

**Proof.** According to proposition 4.9, an inextensional displacement $\mathbf{u}$ can be expressed in the form (4.16) and the derived bending system (4.6)-(4.7) reduces to (4.9). But, as in $\mathbf{X}$ a displacement is reduced to a constant, we have

$$w_1^1(y^1, 0) = \rho_1(y^1) = 0.$$

Consequently:

$$\begin{cases} w_1^1 = w_2^2 = w_2^1 = 0 \\ w_1^2 = \rho_2(y^1), \end{cases} \tag{4.18}$$

in other words $\mathbf{w}_2 = 0$ and $\mathbf{w}_1 = \rho_2(y^1)\mathbf{a}_2$. Therefore, (4.16) reduces to (4.17). □

**Remark 4.11.** We see in (4.15) that on any directrix of a developable surface, i.e. any curve transversal to the generators, the trace of elements of $\mathbf{R}(S)$ make sense in $\mathbf{L}^2$. □

# 5 Geometrical Rigidification of Surfaces by Kinematic Boundary Conditions.

The results of this section are classical, but for sake of completeness, we think useful to recall them as the fundamental role of asymptotic lines of a surface is barely taken into account in the literature of theory of thin shell.

Let us consider a surface $S$ and let $\Gamma$ be a part of its boundary. The fixation or clamping boundary condition on $\Gamma$ implies $\mathbf{u} = 0$ on $\Gamma$. It induces trivially

$$u_1 = u_2 = 0 \text{ on } \Gamma. \tag{5.1}$$

The boundary condition (5.1) can be considered as a Cauchy data for the reduced bending system (3.6) satisfied by any inextensional displacement. Therefore, the uniqueness theorem of Cauchy problem for first order partial differential equation system in two variable‖gives the local inhibition of the surface if $\Gamma$ is transversal to the characteristics of the reduced bending system (3.6), i.e. if $\Gamma$ is transversal to the asymptotic lines of $S$.

**Proposition 5.1.** *Let $S$ be a smooth surface fixed or clamped on a part $\Gamma$ of its boundary. If $\Gamma$ is transverse to the asymptotic lines of the surface, then $S$ is geometrically rigid (or inhibited) in a neighborhood of $\Gamma$.*

Depending on unique continuation theorems for solutions of Cauchy problem (see [15] for the elliptic case), and thus depending on the geometry of $S$, the rigid neighborhood can be precised:

**Proposition 5.2.** *In the framework of proposition 5.1, if $S$ is uniformly elliptic then $S$ is entirely rigid.*

**Proposition 5.3.** *In the framework of proposition 5.1, if $S$ is developable, then $S$ is rigid in the domain $B(\Gamma)$, constituted by the set of the generators of $S$ passing through $\Gamma$.*

In the case of a hyperbolic surface, the rigidified part is determined by the classical uniqueness theorem of Cauchy problem for hyperbolic first order differential system in two variables.

‖Theorem of Holmgren for analytic surfaces or theorem of Carleman for smooth ($C^3$) surfaces, see [13] and [6]

Figure 3: *A partially rigidified developable surface, fixed on* $\Gamma$.

**Proposition 5.4.** *In the framework of proposition 5.1, if S is uniformly hyperbolic then S is rigid in the determination domain issued from* $\Gamma$*, i.e. the region constituted by the set of the points for which the two issued asymptotic lines of S pass through* $\Gamma$*.*

Figure 4: *A partially rigidified hyperbolic surface, fixed on* $\Gamma$.

**Remark 5.5.** The last proposition shows the inhibition in a precise domain. Yet, if the rest of the hyperbolic surface if free of any imposed condition, then S generally admits some non-trivial inextensional displacements. To fix the ideas, let us consider a hyperbolic surface with an asymptotic coordinates map ($\mathbf{r}$,$\Omega$) such that $\Omega$ is a trapeze constituted by a rectangle R and a rectangular triangle T, where with the exception of the hypotenuse $\Gamma$ of T, all the boundary of R and T are asymptotic lines of S.

Figure 5: *The fixation on h induces the inhibition in T but not in R.*

Consequently, if S is fixed on $\Gamma$, then according to last proposition, S is rigid in T. However S is not rigid in R. It is due to the fact, that an arbitrary inextensional displacement on the side $r_1$ of R, it induces a Goursat data for the hyperbolic bending system (see theorem A.2 in Appendix), and thereby determines a non-trivial inextensional displacement on R. □

**Remark 5.6.** Propositions 5.1, 5.2, 5.3 and 5.4 are still valid if, instead of a fixation condition on $\Gamma$, one imposes a rigid displacement on $\Gamma$. Indeed, if a rigid movement $\mathbf{u} = \mathbf{c} \wedge \mathbf{r} + \mathbf{d}$ is imposed on $\Gamma$, it suffices to apply propositions 5.1-5.4 to the displacement: $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{c} \wedge \mathbf{r} + \mathbf{d}$, since one can see that $\tilde{\mathbf{u}}$ vanishes on $\Gamma$. □

As we see the role of asymptotic line is essential in geometrical rigidity. According to last remark, fixing a surface along an asymptotic line is restrictive with respect to the admissible inextensional displacement, but is not restrictive enough to rigidify in general.

Using the framework of the space $\mathbf{R}(S)$, we can give the general expression of the inextensional displacement on non-developable ruled (hyperbolic) surface, fixed or clamped along a generator.

**Proposition 5.7.** *Let $S$ be a non-developable ruled surface given by a map of the form (2.9):*

$$\mathbf{r}(y^1, y^2) = \mathbf{c}(y^1) + y^2\mathbf{g}(y^1).$$

*Let $\sigma$ be the generator at $y^1 = 0$, and let $A(\sigma)$ be the part of $S$ constituted of asymptotic lines of $S$ passing through $\sigma$. If $S$ is fixed or clamped on $\sigma$, then for an inextensional displacement $\mathbf{u}$ on $S$, there is a function $\rho_2 \in L^2$, such that we have in $A(\sigma)$:*

$$\mathbf{u}(y^1, y^2) = \int_0^{y^1} [\rho_2(t)\mathbf{g}(t)] \wedge [\mathbf{c}(y^1) - \mathbf{c}(t) + y^2\mathbf{g}(y^1)]dt. \tag{5.2}$$

**Proof.** Let $\mathbf{u}$ be an inextensional displacement on S, let $\boldsymbol{\omega}$ be its associated rotation field and $\mathrm{R}(\mathbf{u}) = (\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{R}(S)$ ($\mathbf{w}_1 = \boldsymbol{\omega}_{,1}, \mathbf{w}_2 = \boldsymbol{\omega}_{,2}$). We are going to show that $\mathbf{w}_2 = 0$ and $\mathbf{w}_1 = \rho_2(y^1)\mathbf{a}_2$ in $A(\sigma)$.

As $\mathbf{u} \equiv 0$ on $\sigma$ and as $\mathbf{a}_2$ is constant along the generators, we also have:

$$\mathbf{w}_2 \wedge \mathbf{a}_2 = 0 \text{ along } \sigma.$$

In other words, the derivative of $\boldsymbol{\omega}$ along $\sigma$ is tangent to $\sigma$.

Let us put aside for the moment, the map in the form (2.9) and consider an asymptotic coordinates map of $S$, we still denote (abusively) $(\mathbf{r}, \Omega)$, the curve $\sigma$ still being defined at $y^1 = 0$. In such a map, with the simplification:

$$b_{11} = 0, b_{22} = 0 \text{ and } b_{12} \neq 0,$$

and the derived bending system (4.6)-(4.7)becomes:

$$\begin{cases} w^1_{2,1} = -\Gamma^1_{11}w^1_2 + \Gamma^1_{22}w^2_1 \\ w^2_{1,2} = \Gamma^2_{11}w^1_2 - \Gamma^2_{22}w^2_1 \\ w^1_1 = w^2_2 = 0. \end{cases} \tag{5.3}$$

As the derivative of $\boldsymbol{\omega}$ along $\sigma$ is tangent to $\sigma$, i.e. $w^1_2 = 0$ along $\sigma$, the third equation of (5.3) induces then $\mathbf{w}_2 = 0$ along $\sigma$.

On another hand, as S is ruled, the tangents to a generator coincide: $\mathbf{a}_{2,2} = c\mathbf{a}_2$. Therefore we have

$$\Gamma^1_{22} = \mathbf{a}^1.\mathbf{a}_{2,2} = c\mathbf{a}^1.\mathbf{a}_2 = 0.$$

Consequently, the first equation (5.3) becomes

$$w^1_{2,1} = -\Gamma^1_{11} w^1_2.$$

Since $\mathbf{w}_2 = 0$ along $\sigma$, resolving the last ordinary differential equation gives $w^1_2 = 0$ in the domain of $\Omega$ defined with the asymptotic lines issued from $\sigma$, i.e. $A(\sigma)$. Thus, we have showed that $\mathbf{w}_2 = 0$ in $A(\sigma)$.

Let us come back now, to the initial map in the form (2.9). In this map, we have the simplification:

$$b_{22} = \Gamma^1_{22} = \Gamma^2_{22} = 0.$$

Then, the equation (4.7) of the derived bending system, $w^1_1 = w^2_2$, combined with $\mathbf{w}_2 = 0$, gives $w^1_1 = 0$ in $A(\sigma)$.

Consequently, in $A(\sigma)$, the derived bending system (4.6) reduces to:

$$w^2_{1,2} = 0.$$

In other words, $w^2_1 = \rho_1(y^1)$, or $\mathbf{w}_1 = \rho_1(y^1)\mathbf{a}_2$ where $\rho_1(y^1)$ is arbitrary in $L^2$.

Finally, according to proposition 3.12, we obtain the expression (5.2), by replacing in (3.19). □

# 6 Surface with an Edge, Criteria of Constant Angle.

Let § be a surface with an edge along a curve $\Gamma$. It is a surface constituted by two smooth parts $S^+$ and $S^-$, joined together along their common curve $\Gamma$. On every point of the edge $\Gamma$, the tangent planes of the adjacent parts $S^+$ and $S^-$, are supposed to make an angle $\theta$ different from zero and $\pi$).

Figure 6: *A surface with one edge.*

Let $\mathbf{u}$ be a displacement on §. It will be said inextensional if its respective restriction $\mathbf{u}^+$ and $\mathbf{u}^-$ on $S^+$ and $S^-$ are inextensional. In an analogous way, the associated rotation field $\boldsymbol{\omega}$ is defined by the associated rotation field $\boldsymbol{\omega}^+$ and $\boldsymbol{\omega}^-$ to $S^+$ and $S^-$.

Classically, two kind of edge are considered:
• *Simple edge*: the continuity of the displacement is imposed, without any condition on the variation of the angle $\theta$

$$[\mathbf{u}^+ - \mathbf{u}^-] = 0 \text{ along } \Gamma. \tag{6.1}$$

• *Keeping constant angle edge*: the continuity of the displacement is imposed, with the additional condition of the angle of the edge to remain constant in the deformation:

$$[\mathbf{u}^+ - \mathbf{u}^-] = 0 \text{ and } \delta\theta = 0 \text{ along } \Gamma. \tag{6.2}$$

where $\delta\theta$ denotes the linearized variation of the angle $\theta$.
One can notice that the continuity of the displacement does not induce the continuity of the associated rotation field, but we have the following classical results (see [35]):

**Proposition 6.1.** *Let $S$ be a surface with an edge along a curve $\Gamma$. Let $\mathbf{u}$ be an inextensional displacement and $\boldsymbol{\omega}$ its associated rotation field. The jump on $\Gamma$ of $\boldsymbol{\omega}$ is tangent to the edge $\Gamma$. More precisely, we have :*

$$[\boldsymbol{\omega}^+ - \boldsymbol{\omega}^-] = \delta\theta\mathbf{t} \textit{ along } \Gamma, \tag{6.3}$$

*where $\mathbf{t}$ denotes the unit tangential vector to $\Gamma$.*

**Corollary 6.2.** *In the framework of preceding proposition, a displacement $\mathbf{u}$ satisfies the constant angle condition if and only if:*

$$[\boldsymbol{\omega}^+ - \boldsymbol{\omega}^-] = 0. \tag{6.4}$$

Let us introduce now, a new criteria of the constant angle condition, first announced in [8] and expressed for derivatives of rotation field (and therefore for elements of $\mathbf{R}(\S)$, essential afterwards.

**Proposition 6.3.** *Let $S$ be a surface with an edge along a non straight curve $\Gamma$. Let $\boldsymbol{\omega}$ be a rotation field on $S$. If the angle of edge $\theta$ is different from zero and $\pi$, then*
*(i) The curve $\Gamma$ is not an asymptotic line for at least one of the two adjacent part of $S$.*
*(ii) The trace of the derivative of $\boldsymbol{\omega}$ along the edge $\Gamma$ (denoted $D_\Gamma\boldsymbol{\omega}$) make sense in $H^{-\frac{1}{2}}(\Gamma)$ ($L^2(\Gamma)$ in some cases).*
*(iii) If the edge $\Gamma$ is keeping angle constant then $D_\Gamma\boldsymbol{\omega}$ is tangent to $\Gamma$.*

**Proof.** As the edge $\Gamma$ is not a straight line, if $\Gamma$ is an asymptotic line of one of the adjacent part, then the osculating plane of $\Gamma$ coincide with the tangent plane to $S$ on each point of $\Gamma$. Consequently, as the edge makes an angle different from zero and $\pi$, $\Gamma$ cannot be an asymptotic line of both adjacent part. The first assertion is proved. Thus, according to corollary 4.7, the trace of the derivative of $\boldsymbol{\omega}$ along the edge $\Gamma$ make sense.

It is then valid to derive the constant angle condition (6.4) along the edge and we obtain that the jump of the derivative of $\boldsymbol{\omega}$ along the edge vanishes. But, as the derivative of a rotation field is tangent to the surface, $D_\Gamma \boldsymbol{\omega}$ must belong to the intersection of the respective tangent planes of the adjacent parts of §. As these planes make an angle different from zero and $\pi$, on each point of the edge, $D_\Gamma \boldsymbol{\omega}$ is tangent to the edge. □

An interesting case, is the case of an edge keeping angle constant, the curve of which is *plane*. We have the classical property:

**Proposition 6.4.** *Let $S$ be a surface with an edge keeping angle constant along a non straight curve $\Gamma$. If $\Gamma$ is plane, then, up to a rigid displacement, the restriction to $\Gamma$ of an inextensional displacement* $\mathbf{u}$ *is normal to the plane of* $\Gamma$.

**Proof.** We give here a new proof, completing the one of [15].

Let us choose a map $(\mathbf{r}, \Omega)$, such that the curve of edge $\Gamma$ is given at $y^2 = 0$. Let $\boldsymbol{\omega}$ be the rotation field on S, associated to $\mathbf{u}$.

According to the proposition 3.13, we have the explicit expression on $\Gamma$, up to a rigid displacement:

$$\mathbf{u}(y^1, 0) = \int_0^{y^1} \boldsymbol{\omega}_{,1}(t, 0) \wedge [\mathbf{r}(y^1, 0) - \mathbf{r}(t, 0)] dt. \tag{6.5}$$

Let P be the plane of the edge. It is clear that the expression $[\mathbf{r}(y^1, 0) - \mathbf{r}(t, 0)]$ belongs to P, for every $t$. Moreover, according to proposition 6.3, the trace of $\boldsymbol{\omega}_{,1}$ make sense and is tangent to the edge, and *a fortiori* tangent to P. Consequently, the expression (6.5) shows the assertion. □

**Remark 6.5.** In the framework of proposition 6.4, if one of the adjacent parts of §, say $S^+$, is uniformly hyperbolic, then the plane curve of edge is transversal to the asymptotic curves of $S^+$. Indeed, if $\Gamma$ is an asymptotic line of $S^+$, as it is non straight, it would imply that the tangent planes of $S^+$ all along the edge coincide, and therefore that the second fundamental form vanishes, which is opposite with the hyperbolic nature of $S^+$. □

# 7 Geometrical Rigidity of a Surface with an Edge.

In this section, we consider various cases of surface with one edge, considering the (restrictive) effects of the edge on the admissible inextensional displacements. We shall see that the effects are very different, depending on the configuration.

For the sake of simplicity, we will suppose all the smooth parts of the considered surface to be simply connected.

## 7.1 Case of a cone with an edge

**Theorem 7.1.** *Let $S$ be a surface with an edge $\Gamma$ with an angle $\theta$ different from zero and $\pi$. We suppose that one of the adjacent part of $S$ is a cone $S^+$ of summit $X$ (included in $S^+$), for which $\Gamma$ is transversal to the generators. If the edge is keeping constant angle, then $S^+$ is geometrically rigid.*

**Proof.** Let us choose a map of the cone $S^+$ of the form (2.9), such that the edge is defined at $y^2 = const$. Let $\mathbf{u}$ be an inextensional displacement and $\boldsymbol{\omega}$ its associated rotation field. We are going to show that $\boldsymbol{\omega}$ is necessarily constant.

Figure 7: *Two examples of cone with an edge.*

According to (4.18) (see proposition 4.10), we have:

$$\boldsymbol{\omega}_{,1} = \rho(y^1)\mathbf{a}_1 \text{ and } \boldsymbol{\omega}_{,2} = 0.$$

Moreover, according to proposition 6.3, the constant angle condition gives

$$\boldsymbol{\omega}_{,1} \wedge, \mathbf{a}_1 = 0 \text{ along } \Gamma \Longrightarrow \rho(y^1) = 0.$$

Consenquently, $\boldsymbol{\omega}_{,1} = 0$ and $\boldsymbol{\omega}_{,2} = 0$. □

**Remark 7.2.** In the theorem 7.1, the other adjacent part $S^-$ of $S$ is arbitrary. Depending on the geometry of $S^-$, the rigidity extends according to remark 5.6. On the other hand, if $S^-$ is also a cone (containing its summit), it is possible to show, without any condition imposed on the variation of the angle of edge, that $S$ is quasi-rigid, i.e. the space of admissible inextensional is of finite dimension. More precisely, S is inhibited up to a degree of freedom. □

## 7.2 Case of a general developable surface with an edge

**Proposition 7.3.** *Let $S$ be a surface with an edge keeping constant angle on a curve $\Gamma$. Let $S^+$, one of the parts adjacent to $\Gamma$, be developable, and let $\Gamma$ be a directrix of it. Let $\mathbf{u}$ be an inextensional displacement on $S$. For a map of $S^+$ of the form (2.9), there is a function $\rho_1 \in H^1(\Gamma)$ such that, up to a rigid displacement:*

$$\mathbf{u} = \int_0^{y^1} [\rho({y^1}_1)\mathbf{c}'(t)] \wedge [\mathbf{c}(y^1) - \mathbf{c}(t) + y^2 g(y^1)]dt \textit{ on } S^+. \quad \square \tag{7.1}$$

**Proof.** In a map of the form (2.9), as $S^+$ is developable, we have seen in section 4, that every element $(\mathbf{w}_1, \mathbf{w}_2)$ of the corresponding space $\mathbf{R}(\S^+)$ can be written under the form (4.15)

$$\begin{cases} \mathbf{w}_1 = \rho_1(y^1)E(y^1,y^2)\mathbf{a}_1 + [F(y^1,y^2) + \rho_2(y^1)]\mathbf{a}_2 \\ \mathbf{w}_2 = -\rho_1(y^1)E(y^1,y^2)\mathbf{a}_1. \end{cases}$$

On the other hand, according to proposition 6.3, the constant angle condition at the edge gives

$$\mathbf{w}_1 \wedge \mathbf{a}_1 = 0.$$

Replacing in (4.15), we obtain, along the edge:

$$\rho_2(y^1)\mathbf{a}_2 \wedge \mathbf{a}_1(y^1, 0) = 0,$$

which induces $\rho_2 = 0$.
Consequently, replacing in (4.16) of proposition 4.9, we obtain the expression (7.1). □

In the framework of last proposition, the developable surface may indeed admits some non-trivial inextensional displacements.

It is sufficient, for any function $\rho_1$ in (7.1), defining a non-trivial displacement on $S^+$, that the trace on the curve of edge of the associated rotation field, constitutes an admissible Cauchy data for the corresponding derived bending system of $S^-$ (the other adjacent part of $S$) or reciprocally. It is the case, for instance, if $S^-$ is hyperbolic and $\Gamma$ is transversal to its asymptotic lines.

If $S^-$ is also developable, the proposition 7.3 can also be applied on $S^-$. Thus, if no other kinematic condition is imposed on $S$, an arbitrary function $\rho_1$ determines uniquely an admissible inextensional displacement (satisfying the constant angle condition at the edge).

## 7.3 Case of a hyperbolic surface with an edge

**Proposition 7.4.** *Let $S$ be a surface constituted by $S^+$ and $S^-$ joined together along a curve $\Gamma$, making an edge keeping constant angle with angle different from zero and $\pi$. We suppose that both $S^+$ and $S^-$ are hyperbolic and $\Gamma$ is transversal to their respective asymptotic lines. If there is no other condition imposed on S, then the space of inextensional displacement* $\mathbf{G}$ *is of infinite dimension (§ is not rigid).*

**Proof.** As the curve of edge is transversal to the asymptotic lines of $S^+$ and $S^-$, it is a non-characteristic curve for the respective derived bending systems (for any given maps of $S^+$ and $S^-$).

Let us choose for $S^+$ and $S^-$, asymptotic coordinates map (see proposition 2.9). With the simplification (respectively on $S^+$ and $S^-$) of the second fundamental form $b_{11} = b_{22} = 0$, the third equation of the derived bending system (4.6) and (4.7) give (respectively on $S^+$ and $S^-$)

$$w_1^1 = w_2^2 = 0.$$

Let $\phi$ be a function describing the curve $\Gamma = \{(y^1, y^2) \in \Omega / y^2 = \phi(y^1)\}$ (respectively $\phi^+$ and $\phi^-$ on $S^+$ and $S^-$). According to proposition 6.3, the constant angle condition at the edges give (respectively on $S^+$ and $S^-$)

$$[\mathbf{w}_1 + \phi'\mathbf{w}_2] \wedge [\mathbf{a}_1 + \phi'\mathbf{a}_2] = 0. \tag{7.2}$$

In other words

$$w_1^2(y^1, \phi(y^1)) = \phi'^2 w_2^1(y^1, \phi(y^1)). \tag{7.3}$$

Thus, any arbitrary function in $L^2(\Gamma)$, will give respectively on $S^+$ and $S^-$ an admissible Cauchy data for the respective derived bending system of $S^+$ and $S^-$, and thereby determines an inextensional displacement on §, according to classical existence and uniqueness theorem of the Cauchy problem for hyperbolic system in two variables. □

**Remark 7.5.** An interesting case, where all the conditions of last proposition are satisfied, is that of two hyperbolic surfaces joined along a non-straight curve which is *plane*, see remark 6.5. □

**Remark 7.6.** The proposition 7.4 remains true, if $S^-$ is developable, see above, the case of developable surface with an edge. □

**Remark 7.7.** In the proposition 7.4 and last remark, the hypothesis that the curve $\Gamma$ of edge is transversal to the asymptotic lines, is essential. Indeed, we shall see in next section (theorem 8.2), that if $\Gamma$ is an asymptotic line, and non straight, then the edge rigidify, and moreover rigidify locally (at least) $S^-$ whatever is its geometrical nature. □

### 7.4 Case of an elliptic surface with an edge

Some results concerning elliptic surfaces with an edge can be found in [35] or even [15], for which we refer. Netherveless, we would like to point out one result:

**Theorem 7.8.** *Let $S$ be an elliptic surface with boundary $\Gamma$ joined together with another arbitrary surface all along $\Gamma$, making an angle different from zero and $\pi$. If the edge is keeping the angle constant, then $S$ is rigid.*

The principle of proof of theorem 7.8 is based on theory of generalized analytic functions for which the constant angle criteria appears as a generalized Riemman-Hilbert Problem. The rigidity follows then from the uniqueness and unique continuation of a solution of such problem, see [35].

## 8 A Theorem of Rigid Edge.

We are now considering one of our new results which is a rigidity property of an edge when the curve of the edge is an asymptotic line. It emphasizes the fundamental role of asymptotic lines in theory of inextensional displacements and thereby in theory of thin elastic shells.

To have an intuitive idea of the result we can think first of the case of a straight edge. The rigidity of straight edge made by a folded plane is familiar, we can think of a folded sheet of paper.

**Theorem 8.1.** *Let $S$ be a surface with a simple edge making an angle different from zero and $\pi$ (no condition is imposed on the variation of the angle of edge). If the curve of edge is a straight line, then the edge is rigid.*

We refer to [15] for a formal proof of theorem 8.1. Actually, it is possible, in the framework of this paper and using the space $\mathbf{R}(S)$, to give a rigorous proof, but we shall not do it here.

**Theorem 8.2.** *Let $S$ be a surface with an edge (with angle different from zero and $\pi$) along a curve $\Gamma$. We suppose that the edge is keeping angle constant, and one of the adjacent part $S^+$ of $S$ is hyperbolic in a neighborhood of the edge; no hypothesis is made on $S^-$. If $\Gamma$ is an asymptotic line of $S^+$, then the edge $\Gamma$ is rigid.*

**Proof.** The theorem 8.1 proves the rigidity of the curve of edge if it a straight line. Thus, we can only consider the case where $\Gamma$ has a non-zero curvature.

Let us choose for $S^+$ an asymptotic coordinates map, in which the coordinates curves are the asymptotic lines of $S$; we can suppose without any loss of generality that $\Gamma$ is defined at $y^2 = 0$. In such a map, the coefficients of the second fundamental form simplify (see proposition 2.9)

$$b_{11} = b_{22} = 0.$$

The third and fourth equations of the derived bending system (4.6)-(4.7) give (respectively on $S^+$ and $S^-$)

$$w_1^1 = w_2^2 = 0.$$

On another hand, the constant angle condition at the edge gives (see proposition 6.3)

$$\mathbf{w}_1 \wedge \mathbf{a}_1 = 0 \implies w_1^2 = 0,$$

in other words

$$\mathbf{w}_1 = 0 \text{ along } \Gamma.$$

Consequently, according to the proposition 3.16, the edge $\Gamma$ is rigid. □

This result completes the proposition 7.4, where we saw that in the case of an edge transversal to the asymptotic lines, the rigidity property is totally different.

**Remark 8.3.** In the framework of theorem 8.2, if $\Gamma$ is not a straight line, it cannot be a plane curve, since this should be in contradiction with the hyperbolic nature of $S^+$, see remark 6.5. □

**Remark 8.4.** In the framework of theorem 8.2, if $\Gamma$ is not a straight line, it cannot also be an asymptotic line of $S^-$. Indeed, as the respective tangent plane to $S^+$ and $S^-$ would coincide with the osculating plane of $\Gamma$, this should be in contradiction with the hypothesis that the angle of edge is different from 0 and $\pi$. □

According to last remark, in the framework of theorem 8.2, if $\Gamma$ is not a straight line, it is then transversal to the possible asymptotic lines of $S^-$. Therefore, as $\Gamma$ is rigid, according to remark 5.6 and proposition 5.1, we have immediately:

**Corollary 8.5.** *In the framework of theorem 8.2, if the curve of edge has a non-zero curvature and if $S^-$ is smooth enough, then $S^-$ rigid in a neighborhood of the edge.*

**Remark 8.6.** Of course, depending on the nature of $S^-$, the rigidified neighborhood can be specified. For instance if $S^-$ is uniformly elliptic, $S^-$ is entirely rigid, see the remark 5.6 and the proposition 5.2. □

**Remark 8.7.** We must keep in mind that if no other conditions are imposed (as kinematic boundary conditions), the hyperbolic part is not rigid. This is due to the fact that the rigidity of the edge induces a Cauchy data for the bending system of $S^+$, but on a characteristic curve. As this corresponding bending system is of hyperbolic nature, the existence and uniqueness theorem for the Goursat problem proves the existence of admissible inextensional displacements on $S$, see remark 5.5. □

The framework of theorem 8.2 is remarkable, as it shows an example of surface with an edge, geometrically rigid on one side of the edge and *not* rigid on the other side.

# 9 Case of Rigidification by Two Edges Keeping Constant Angle.

We saw, in the preceding section that, an edge was restrictive with respect to the inextensional displacements. But, as we saw in some examples, some non trivial inextensional displacements generally remain admissible on surface with an edge. But, we can expect that the combinate effects of several edges could be "rigidifying", when one alone was "insufficient".

We consider first some cases of ruled surfaces (developable case and non-developable case), first announced in [9]. Then, we shall consider general hyperbolic surfaces with two edges.

In the sequel, we say a surface $S$ admits an edge along a curve $\Gamma$, if $S$ is joined along $\Gamma$, with another smooth ($C^2$) surface, *a priori* unspecified, making an angle different from 0 and $\pi$.

## 9.1 Case of a ruled surface admitting two edges

Let $S$ be a ruled surface, defined by a map as in (2.9):

$$\mathbf{r}(y^1, y^2) = \mathbf{c}(y^1) + y^2\mathbf{g}(y^1).$$

The straight lines defined at $y^1 = const.$ are the generators, and the coordinates curve defined at $y^2 = const.$ are the directrices.

As the developable case and non-developable cases are of different nature, we consider them separately.

**Theorem 9.1.** *Let S be a developable surface admitting two edges along curves $\Gamma_1$ and $\Gamma_2$, transversal to the generators of S. For sake of simplicity, we suppose that the boundary of S is constituted by generators and the curves $\Gamma_1$ and $\Gamma_2$. If the edges are keeping the angle constant, then S is quasi-rigid. More precisely, S is rigid up to one degree of freedom, at most.*

**Proof.** Let us choose a map as in (2.9), $\mathbf{r}(y^1, y^2) = \mathbf{c}(y^1) + y^2\mathbf{g}(y^1)$, such that the edge $\Gamma_1$ is defined as the coordinate curve $y^2 = 0$. According to proposition 7.3, there is a function $\rho_1 \in H^1(\Gamma_1)$ such that an inextensional displacement $\mathbf{u}$ can be expressed in the form (7.1):

$$\mathbf{u} = \int_0^{y^1} [\rho(y^1{}_1)\mathbf{c}'(t)] \wedge [\mathbf{c}(y^1) - \mathbf{c}(t) + y^2 g(y^1)]dt \text{ on } S^+.$$

In particular, for any couple $(\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{R}(S)$, we have

$$\begin{cases} \mathbf{w}_1 = \rho_1(y^1)E(y^1, y^2)\mathbf{a}_1 + F(y^1, y^2)\mathbf{a}_2 \\ \mathbf{w}_2 = -\rho_1(y^1)E(y^1, y^2)\mathbf{a}_2, \end{cases}$$

where

$$\begin{cases} E(y^1, y^2) = \exp[-\int_0^{y^1} 2\Gamma_{12}^1(y^1, s)ds] \\ F(y^1, y^2) = \int_0^{y^2} \left[[\rho_1 E(y^1, s)]_{,1} - 2\Gamma_{12}^2\rho_1 E(y^1, s)\right] ds. \end{cases}$$

Let us consider the second edge on $\Gamma_2$, we can suppose it is defined by a function $\psi$:

$$\Gamma_2 = \{(y^1, y^2) \in \Omega / y^2 = \psi(y^1)\}.$$

According to proposition 6.3, the constant angle condition imposes, along $\Gamma_2$:

$$(\mathbf{w}_1 + \psi'\mathbf{w}_2) \wedge (\mathbf{a}_1 + \psi'\mathbf{a}_2) = 0,$$

i.e.

$$\psi' w_1^1(\mathbf{a}_1 \wedge \mathbf{a}_2) + w_1^2(\mathbf{a}_2 \wedge \mathbf{a}_1) + \psi' w_2^2(\mathbf{a}_1 \wedge \mathbf{a}_2) = 0.$$

Thus, since $w_1^1 + w_2^2 = 0$, we have

$$2\psi' w_1^1 + w_1^2 = 0.$$

This induces an ordinary differential equation satisfied by the function $\rho_1$:

$$a(y^1)\rho_1(y^1) + b(y^1)\rho_1'(y^1) = 0, \tag{9.1}$$

where the functions $a$ and $b$ are given by

$$\begin{cases} a(y^1) = 2\psi'(y^1)E(y^1, \psi(y^1)) + \int_0^{\psi(y^1)}[E_{,1}((y^1, s) - 2\Gamma_{12}^2 E(y^1, s)]ds \\ b(y^1) = \int_0^{\psi(y^1)} E(y^1, s)]ds. \end{cases}$$

Then the function $\rho_1$, and the admissible inextensional displacement $\mathbf{u}$ are determined, up to a degree of freedom. $\square$

**Remark 9.2.** In the framework of theorem 9.1, the developable surface is, in many cases, actually rigid. It depends on the geometrical nature of the adjacent surface. For instance, if one of the adjacent surface is hyperbolic, for which the curve of edge is (possibly partly) an asymptotic line, see theorem 8.2. $\square$

Let us consider the non-developable (hyperbolic) case.

**Theorem 9.3.** *Let $S$ be a non-developable ruled surface, admitting an edge keeping angle constant along a curve $\Gamma$ and a second edge along a portion of generator $\sigma$ (the angle of edge on $\sigma$ can vary). Let $A(\sigma)$ be the part of $S$ constituted by the asymptotic lines of $S$ passing through $\sigma$, and let $B(\Gamma)$ be the part of $S$ constituted by the generators at $y^1 = const.$ passing through $\Gamma$. We suppose moroever that $\Gamma \subset A(\sigma)$, then the part $D = A(\sigma) \cap B(\Gamma)$ is rigid.*

Figure 8: *Different configurations of theorem 9.3*

**Proof.** Let us choose a map of $S$ in the form (2.9): $r(y^1, y^2) = \mathbf{c}(y^1) + y^2\mathbf{g}(y^1)$ such that, $\Gamma$ is the directrix at $y^2 = 0$, and $\sigma$ is defined at $y^1 = 0$.

Let $(\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{R}(S)$. According to theorem 8.1, and theorem 5.7, the edge $\sigma$ is rigid and there is a function $\rho \in L^2(\Gamma)$ such that we have on $A(\sigma)$:

$$\mathbf{w}_1(y^1, y^2) = \rho(y^1)\mathbf{a}_2$$
$$\mathbf{w}_2(y^1, y^2) = 0.$$

Consider now, the edge on $\Gamma$. According to proposition 6.3, the constant angle condition imposes:

$$\mathbf{w}_1(y^1, 0) \wedge \mathbf{a}_1 = 0.$$

Consequently $\rho \equiv 0$ in $B(\Gamma)$, and therefore $(\mathbf{w}_1, \mathbf{w}_2) = 0$ in $A(\sigma) \cap B(\Gamma)$. $\square$

**Remark 9.4.** In theorem 9.3, the two curves of edges $\sigma$ and $\Gamma$ does not necessarily intersect, see figure 8.a. On the other hand, in some configuration it possible that the two curves of edges intersect, but the rigidified domain is empty, see figure 8.b. In the case where $\Gamma$ is not entirely included in $A(\sigma)$, $\Gamma \cap A(\sigma) = \Gamma_0$, the theorem 9.3 remain valid in the corresponding domain $A(\sigma) \cap B(\Gamma_0)$. $\square$

## 9.2 Cases of general hyperbolic surfaces with two edges

We shall prove a new rigidity result, which can be considered as a generalization of theorem 9.3, although little different as we shall suppose that the two curves of edges join together at a point. We shall use in an essential way, a variant of Goursat problem for hyperbolic systems, which we shall develop in Appendix (theorem A.2).

The only other essential hypothesis we shall make, is to suppose there is no asymptotic line issued from the point where the edges join together. We call it the PLASP condition.

Figure 9: PLASP condition in asymptotic coordinates map.

We first see a special case, when the curves of edges are asymptotic lines:

**Theorem 9.5.** *Let $S$ be a hyperbolic surface admitting two edges on the curves $\Gamma_1$ and $\Gamma_2$, joined together on the point $M$ and satisfying the PLASP condition. Moreover, we suppose that the edges are keeping angle constant, which are different from zero and $\pi$. If the curves $\Gamma_1$ and $\Gamma_2$ are asymptotic lines of $S$, then $S$ is rigid in the determination domain defined by $\Gamma_1$ and $\Gamma_2$.*

**Proof.** Let us choose an asymptotic coordinate map for $S$. Let $(\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{W}$. With the simplification $b_{11} = b_{22} = 0$, the derived bending system (4.6)-(4.7) reduces to:

$$\left\{ \begin{array}{l} w^1_{2,1} = -\Gamma^1_{11} w^1_2 + \Gamma^1_{22} w^2_1 \\ w^2_{1,2} = \Gamma^2_{11} w^1_2 - \Gamma^2_{22} w^2_1 \\ w^1_1 = w^2_2 = 0. \end{array} \right. \tag{9.2}$$

We see that first and second equation of (9.2) is a hyperbolic partial differential system in diagonal form, the characteristics of which are the coordinates curves.

As the curves $\Gamma_1$ and $\Gamma_2$, are asymptotic lines we can suppose, without loss of generality that they are respectively defined on $y^2 = 0$ and $y^1 = 0$.

According to the theorem 8.2, the constant angle condition induces that the curves $\Gamma_1$ and $\Gamma_2$ are rigid. This implies that :

$$w^2_1(y^1, 0) = 0 \text{ and } w^1_2(0, y^2) = 0.$$

In other words, the constant angle condition on the two edges gives a degenerate Goursat data for the derived bending system (9.2).

The PLASP condition insures us the possibility to suppose that the point of $S$ are only defined with $y^2 \geq 0$ and $y^1 \geq 0$. Therefore the classical uniqueness of degenerate Goursat problem for hyperbolic system induces that $(\mathbf{w}_1, \mathbf{w}_2) = (0, 0)$, i.e. $S$ is rigid in the determination domain defined by the edges. □

Actually, the hypothesis of the curves of the edges to be asymptotic lines is not essential.

**Theorem 9.6.** *In the framework of theorem 9.5, we suppose that $\Gamma_1$, one of the two keeping angle constant edges, is an asymptotic line of S and the other edge $\Gamma_2$ is transversal to the asymptotic lines of S. If the PLASP condition is satisfied by $\Gamma_1$ and $\Gamma_2$ in S, then S is rigid in the determination domain of $\Gamma_2$.*

**Proof.** Let us choose an asymptotic coordinate map for $S$. Let $(\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{R}(S)$. In this map the derived bending system (4.6)-(4.7) reduces to (9.2), which is a hyperbolic first order P.D.E system in diagonal form. As $\Gamma_1$ is an asymptotic line of S and $\Gamma_2$ is transversal to the asymptotic lines of S, we can suppose, without loss of generality, that they are respectively defined at $y^2 = 0$ and $y^1 = \phi_2(y^2)$. According to theorem 8.2 and proposition 6.3, the constant angle condition then induces:

$$\begin{cases} \mathbf{w}_1(y^1, 0) = 0 \text{ along } \Gamma_1 \\ \left[\phi_2'(y^2)\mathbf{w}_1 + \mathbf{w}_2\right] \wedge \left[\phi_2'(y^2)\mathbf{a}_1 + \mathbf{a}_2\right] = 0 \text{ along } \Gamma_2. \end{cases}$$

In other words:

$$\begin{cases} w_1^2(y^1, 0) = 0 \\ w_2^1(\phi_2(y^2), y^2) = [\phi_2'(y^2)]^2 w_1^2(\phi_2(y^2), y^2). \end{cases} \tag{9.3}$$

The PLASP condition insures that the points of S can defined by :

$$\{y^1 \geq 0 \text{ and } \phi_2 \geq y^2 \geq 0\}.$$

The problem (9.2),(9.3) is then a non-classical *variant of Goursat problem*, the existence and unicity theorem of which is be proved in Appendix, theorem A.2. It gives that the unique solution to (9.2),(9.3) is the null solution. □

Last we see in this section, that the hypothesis of one of the edges to be asymptotic lines is not essential in theorem 9.6, provided the PLASP condition is satisfied :

**Theorem 9.7.** *In the framework of theorem 9.6, if the curves of edges are transversal to the asymptotic line of S and if the edges $\Gamma_1$ and $\Gamma_2$ join on the point M, satisfying the PLASP condition and making angles different from zero, then S is rigid in a neighborhood of the point M.*

**Proof.** Let us take again an asymptotic coordinates map for S. For any $(\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{R}(S)$, the derived bending system (4.6)-(4.7) reduces to (9.2), which is a hyperbolic first order P.D.E system in diagonal form.

Let $\Gamma_1$ and $\Gamma_2$ be the curves of edge on S, and $\phi_1$ and $\phi_2$ be two functions describing them :

$$\Gamma_1 = \{(y^1, y^2)/y^2 = \phi_1(y^1)\} \text{ and } \Gamma_2 = \{(y^1, y^2)/y^1 = \phi_2(y^2)\}$$

As the curves $\Gamma_1$ and $\Gamma_2$ are supposed to be transversal to the asymptotic lines, the functions $\phi_1$ and $\phi_2$ are invertible. In other words their first derivatives do not vanish.

Let us choose M$= (0,0)$, so that we have $\phi_1(0) = 0$ and $\phi_2(0) = 0$.

In this framework, according to proposition 6.3, the constant angle condition on the edges gives :

$$\left\{ \begin{array}{l} w_1^2(y^1, \phi_1(y^1) = [\phi_1'(y^1)]^2 w_2^1(y^1, \phi_1(y^1)) \text{ on } \Gamma_1 \\ w_2^1(\phi_2(y^2), y^2) = [\phi_2'(y^2)]^2 w_1^2(\phi_2(y^2), y^2) \text{ on } \Gamma_2. \end{array} \right. \tag{9.4}$$

We shall show now the uniqueness of the problem (9.2),(9.4) in a neighborhood of the point (0,0), which amounts to prove the rigidity of $S$ in a neighborhood of the point M.

Let $\Omega_T = [0, T_1] \times [0, T_2]$ be a set defining a neighborhood of the point M.

As we have supposed the PLASP condition, we can choose the orientation of the parameters such that

$$\left\{ \begin{array}{l} \phi_1(y^1) \geq 0 \text{ in } [0, T_1] \\ \phi_2(y^2) \geq 0 \text{ in } [0, T_2]. \end{array} \right. \tag{9.5}$$

We can suppose furthermore, without loss of generality, that the two functions $\phi_1$ and $\phi_2$ are invertible, in a neighborhood of 0.

Let $A$ be the unbounded operator defined from :

$$D(A) = \{(f_1, f_2) \in \left[L^2(\Omega_T)\right]^2 / f_{1,1} \in L^2(\Omega_T) \text{ and } f_{2,2} \in L^2(\Omega_T)\}$$

with value in $\left[L^2(\Omega_T)\right]^2$, given by $A(f_1, f_2) = (g_1, g_2)$, such as :

$$\left\{ \begin{array}{l} g_1(y^1, y^2) = \int_{\phi_2(y^2)}^{y^1} [-\Gamma_{11}^1 f_1 + \Gamma_{22}^1 f_2] d\hat{y}^1 + [\phi_2'(y^2)]^2 f_2(\phi_2(y^2), y^2) \\ g_2(y^1, y^2) = \int_{\phi_1(y^1)}^{y^2} [\Gamma_{11}^2 f_1 - \Gamma_{22}^2 f_2] d\hat{y}^2 + [\phi_1'(y^1)]^2 g_1(y^1, \phi_1(y^1)). \end{array} \right. \tag{9.6}$$

where the integral operators in (9.6) are defined as the continuous extension in $L^2$ of integral operator defined for continuous functions.

It is clear that a solution of the problem (9.2)-(9.4) is a fixed point of the operator $A$. We then show that $A$ has a unique fixed point in $\Omega_T$.

Let $(g_1, g_2)$ be a fix point of the operator $A$. From (9.6), we easily obtain the estimates, with a positive constant $C$ which only depends on coefficients of the surface $S$, not necessarily the same at each occurrence :[**]

$$\begin{array}{rcl} \|g_1\|_{L^2(\Omega_T)}^2 & \leq & CT_1 \left[ \|g_1\|_{L^2(\Omega_T)}^2 + \|g_2\|_{L^2(\Omega_T)}^2 \right] \\ & + & T_1 \left[ \int_0^{T_2} \left| \phi_2'^2(s) g_2(\phi_2(s), s) \right|^2 ds \right]. \end{array} \tag{9.6}$$

[**] All the following estimates are satisfied for $C = \sup_\Omega \{|\Gamma_{\alpha\beta}^\lambda|\}$.

But, according to lemma A.1, we have

$$\left[\int_0^{T_2} \left|\phi_2'^2(s) g_2(\phi_2(s), s)\right|^2 ds\right] \leq |\phi_2'| \left[\frac{\|g_2\|^2_{L^2(\Omega_T)}}{T_2} + \|g_{2,2}\|^2_{L^2(\Omega_T)}\right].$$

Furthermore, as $(g_1, g_2)$ is a solution of the system (9.2), we have

$$\|g_{2,2}\|^2_{L^2(\Omega_T)} \leq C\left[\|g_1\|_{L^2(\Omega_T)} + \|g_2\|_{L^2(\Omega_T)}\right].$$

Thus, the inequality (9.6) becomes:

$$\begin{aligned} \|g_1\|^2_{L^2(\Omega_T)} &\leq CT_1\left[\|g_1\|^2_{L^2(\Omega_T)} + \|g_2\|^2_{L^2(\Omega_T)}\right] \\ &+ |\phi_2'|\frac{T_1}{T_2}\left[\|g_2\|^2_{L^2(\Omega_T)} + CT_1\left[\|g_1\|^2_{L^2(\Omega_T)} + \|g_2\|^2_{L^2(\Omega_T)}\right]\right]. \end{aligned}$$

Analogously

$$\begin{aligned} \|g_2\|^2_{L^2(\Omega_T)} &\leq CT_2\left[\|g_1\|^2_{L^2(\Omega_T)} + \|g_2\|^2_{L^2(\Omega_T)}\right] \\ &+ |\phi_2'|\frac{T_2}{T_1}\left[\|g_2\|^2_{L^2(\Omega_T)} + CT_2\left[\|g_1\|^2_{L^2(\Omega_T)} + \|g_2\|^2_{L^2(\Omega_T)}\right]\right], \end{aligned}$$

so that:

$$\|g_2\|^2_{L^2(\Omega_T)} \leq CT_2\left[\|g_1\|_{L^2(\Omega_T)} + \|g_2\|^2_{L^2(\Omega_T)}\right] + |\phi_2'||\phi_1'|\|g_2\|^2_{L^2(\Omega_T)}$$

or

$$\|g_2\|^2_{L^2(\Omega_T)} \leq 2CT_2\left[\|g_1\|_{L^2(\Omega_T)} + \|g_2\|^2_{L^2(\Omega_T)}\right] + \beta\|g_2\|^2_{L^2(\Omega_T)}. \tag{9.7}$$

Analogously

$$\|g_1\|^2_{L^2(\Omega_T)} \leq 2CT_1\left[\|g_1\|_{L^2(\Omega_T)} + \|g_2\|^2_{L^2(\Omega_T)}\right] + \beta\|g_1\|^2_{L^2(\Omega_T)}, \tag{9.8}$$

where

$$\beta = \sup_{y^1\in[0,T_1]} |\phi_1'(y^1)| \times \sup_{y^2\in[0,T_2]} |\phi_2'(y^2)|.$$

One can notice that in the case where at least one of the edges is an asymptotic line of $S$, we would have $\beta = 0$.

Let us denote now by $p_1$ and $p_2$, the slope of the curves defined by the functions $\phi_1$ and $\phi_2$ at the point 0, and let $p$ be their product:

$$\begin{aligned} p_1 &= \phi_1'(0) \\ p_2 &= \phi_2'(0) \\ p &= p_1 \times p_2. \end{aligned}$$

As the two functions $\phi_1$ and $\phi_2$ are invertible, $p$ is different from 0. But, it is moreover different from 1. Indeed, if $p = 1$, it would mean that $p_1 = p_2^{-1}$, in other words that the edges join on M with a null angle.

Thus, if the chosen functions $\phi_1$ and $\phi_2$ give $p > 1$, it suffices to choose alternative functions describing the curves of edges: $\tilde{\phi_1}(y^1) = \phi_2^{-1}(y^1)$ and $\tilde{\phi_2}(y^2) = \phi_1^{-1}(y^2)$, in order to have the corresponding $p < 1$.

Figure 10: *Extension of the rigid domain in theorem 9.7.*

Consequently, for some $T_1$ and $T_2$ sufficiently small, we obtain $\beta < 1$. That is to say, for sufficiently small $T_1$ and $T_2$, the inequalities (9.7) and (9.8) are absurd unless $g_1 = g_2 = 0$. $\square$

**Remark 9.8.** It is clear, in the configuration of theorem 9.7, that the rigidification does not only hold in a small neighborhood of the point M. Depending on the geometry of the curves of edges, the rigidification extends step by step. Furthermore, according to theorem 9.6, the curves of edges are not necessarily totally transversal to the asymptotic lines. $\square$

# 10 Application to Thin Linear Elastic Shells – A New Example of Sensitive Shell

We have recalled in the introduction of this paper that the asymptotic behavior of a thin elastic shell, the thickness of which tends to zero, depends essentially on its geometrical rigidity. In this last section we give a more detailed overview of this theory (we refer to [29], [30], [31],and [11]), where we see how some numerical difficulties may appear in the study of very thin shells, such as membrane locking in the case of non-inhibited shells and sensitive problem in the case of not-well-inhibited shells. In the inhibited case, by applying the rigidity results of the former section, we present a new example of sensitive shell, consisting of a hyperbolic mean surface with two constant angle edges satisfying the PLASP condition.

**V** still denotes the space of admissible displacements, and **G** denotes the space of admissible inextensional displacements (also called infinitesimal bendings).

In the framework of Koiter's *linear bidimensionnal* model of thin elastic shells, the deformation energy bilinear form $a^e$ can be broken down into two separate bilinear forms (see [4] or [29]):

$$a^e = e^3 a^f + e a^m, \tag{10.1}$$

where $e^3 a^f$ denotes the *bending deformation* energy bilinear form and $ea^m$ denotes the *membrane deformation* energy bilinear form. Both $a^f$ and $a^m$ are independent of the thickness $e$.

If the coefficients of membrane elasticity of the shell are denoted by $A^{\alpha\beta\lambda\mu}$, we have:

$$\begin{array}{l} a^m(\mathbf{u},\mathbf{v}) = \int_S A^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\mathbf{u})\gamma_{\lambda\mu}(\mathbf{v})ds \\ a^f(\mathbf{u},\mathbf{v}) = \int_S A^{\alpha\beta\lambda\mu}\rho_{\alpha\beta}(\mathbf{u})\rho_{\lambda\mu}(\mathbf{v})ds, \end{array} \tag{10.2}$$

where $\gamma_{\alpha\beta}$ is the linear variation of the first fundamental form, and $\rho_{\alpha\beta}$ is the linear variation of the second fundamental form.

For each fixed thickness $e > 0$, the bilinear form $a^e$ is continuous and *coercive* in the Sobolev space $H^1 \times H^1 \times H^2$ (tangential components $\times$ normal component) with a clamping condition on a part of the boundary, [†‡]see [4]. Applying the Lax-Milgram theorem, we have :

*For every fixed thickness $e > 0$, the static mechanical problem :*

$$\left\{ \begin{array}{l} \text{Find a displacement } \mathbf{u} \in \mathbf{V} \text{ such that} \\ a^e(\mathbf{u},\mathbf{v}) = (\mathbf{F},\mathbf{v}) \ \forall \mathbf{v} \in \mathbf{V}, \end{array} \right. \tag{10.3}$$

*has a unique solution for each given force $\mathbf{F} \in \mathbf{V}'$ applied on the shell.*

Actually, the space $\mathbf{G}$ of inextensional displacements coincides with the kernel of the membrane deformation energy bilinear form $a^m$, as by definition a displacement $\mathbf{u}$ is inextensional if $\gamma_{\alpha\beta}(\mathbf{u}) = 0$, see (3.3). It is the reason why, the asymptotic behavior will depend on whether the mean surface of the shell is rigid or not.

## 10.1 Non-inhibited shell and membrane numerical locking

In the case of a non-geometrically-rigid surface (non-inhibited shell), the limit behavior takes place in the space $\mathbf{G}$ which is a closed subspace of $\mathbf{V}$. It can be seen by laying down the following scaling, with $\mathbf{f}$ independent of the thickness $e$:

$$\mathbf{u} = \mathbf{u}^\epsilon, \ \mathbf{F} = e^3\mathbf{f} \text{ and } \epsilon = e^2. \tag{10.4}$$

The problem (10.3) can be rewritten as :

$$\left\{ \begin{array}{l} \text{Find a displacement } \mathbf{u}^\epsilon \in \mathbf{V} \text{ such that :} \\ \epsilon^{-1}a^m(\mathbf{u}^\epsilon,\mathbf{v}) + a^f(\mathbf{u}^\epsilon,\mathbf{v}) = (\mathbf{f},\mathbf{v}) \ \forall \mathbf{v} \in \mathbf{V}. \end{array} \right. \tag{10.5}$$

When $e$ (or $\epsilon$) tends to zero, the problem (10.5) can be considered as a penalty problem : the membrane deformations are penalized by the coefficient $\epsilon^{-1}$ compared

[†‡]Without such kinematic boundary conditions, the coercivenes of $a^e$ is established in $\mathbf{V}$ quotiented by the set of admissible rigid displacement.

to the bending deformations. This is quite coherent with our intuition, one can think of a sheet of paper which is "difficult" to stretch, but easy to bend. The deformation tends to go to the *chasm* of minimization of energy : the kernel of the bilinear form $a^m$, i.e. the space $\mathbf{G}$ of inextensional displacements. Classically, see [31], we have:

*For each $\epsilon > 0$, let $\mathbf{u}^\epsilon$ be the unique solution of problem (10.5), then*

$$\mathbf{u}^\epsilon \longrightarrow \mathbf{u}^o \text{ in } \mathbf{V}, \ (strongly)$$

*where $\mathbf{u}^o$ is the solution of :*

$$\left\{ \begin{array}{l} \text{Find a displacement } \mathbf{u}^o \in \mathbf{G} \text{ such that :} \\ a^f(\mathbf{u}^o, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \ \forall \mathbf{v} \in \mathbf{G}. \end{array} \right. \tag{10.6}$$

In other words, with a non-geometrically rigid surface (i.e. a non-inhibited shell), it appears some important deformations for "very small" applied external forces[‡‡]It means a *weakness* of the mechanical structure, which can go *unnoticed* by numerical computations in case where the *membrane numerical locking* phenomenon occurs.

It can be understoodi, as in numerical studies (with exact integration schemes for sake of simplicity), the continuous problem is replaced by a discretized, finite dimensional one:

$$\left\{ \begin{array}{l} \text{Find a displacement } \mathbf{u}_h^\epsilon \in \mathbf{V}_h \text{ such that :} \\ \epsilon^{-1} a^m(\mathbf{u}_h^\epsilon, \mathbf{v}) + a^f(\mathbf{u}_h^\epsilon, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \ \forall \mathbf{v} \in \mathbf{V}_h, \end{array} \right. \tag{10.7}$$

where $\mathbf{V}_h$ denote the finite dimension discretization of $\mathbf{V}$ with the mesh step $h$.

By analogy with the continuous problem, we have the following convergence result:

*For each $\epsilon > 0$, let $\mathbf{u}_h^\epsilon$ be the unique solution of problem (10.7), then*

$$\mathbf{u}_h^\epsilon \longrightarrow \mathbf{u}_h^o \text{ in } \mathbf{V} \ (strongly),$$

*where $\mathbf{u}_h^o$ is the solution of :*

$$\left\{ \begin{array}{l} \text{Find a displacement } \mathbf{u}_h^o \in \mathbf{G}_h \text{ such that :} \\ a^f(\mathbf{u}^o, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \ \forall \mathbf{v} \in \mathbf{G}_h, \end{array} \right. \tag{10.8}$$

where $\mathbf{G}_h = \{\text{inextensional displacements}\} \cap \mathbf{V}_h$.

In fact, in usual finite elements methods, if the mesh size is *small enough*, one will obtain a good approximation of the exact solution of the mechanical problem (10.3) as prove the convergence theorems of various finite element methods, see [3]. But, it appears that the *mesh step has to be as small as the thickness.*

Since computers are limited in speed and memory, in practice, the mesh step is sometime taken too big with respect to i the appropriate mesh for *very* thin elastic shells.

[‡‡]very small" force in the sense of the scaling (10.4).

Actually, the usual finite element methods involve polynomials, and it is shown, in some case, that $\mathbf{G}_h$ contains no polynomial but zero and whereas some quasi-inextensional displacements are solutions of problem (10.3), the numerical computation exhibits a nearly zero solution, see [10] and [7] for a further discussion on this phenomenon.

## 10.2 Inhibited shell and sensitive problem

In the case of a inhibited shell, i.e. its mean surface is geometrically rigid, laying down the scaling :

$$\mathbf{u} = \mathbf{u}^{\epsilon},\ \mathbf{F} = e\mathbf{f},\ \text{and}\ \epsilon = e^2, \tag{10.9}$$

the function $\mathbf{f}$ being independent of $e$, the problem (10.3) can be rewritten as:

$$\left\{ \begin{array}{l} \text{Find a displacement } \mathbf{u}^{\epsilon} \in \mathbf{V} \text{ such that :} \\ a^m(\mathbf{u}^{\epsilon}, \mathbf{v}) + \epsilon a^f(\mathbf{u}^{\epsilon}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})\ \forall \mathbf{v} \in \mathbf{V}. \end{array} \right. \tag{10.10}$$

We note that the bending term $\epsilon a^f$, disappears when $\epsilon \to 0$. In fact, some *second order* partial derivatives of the displacement occur in $a^f$, whereas it does not in $a^m$. The problem (10.10) is then a *singular perturbation* problem when $\epsilon \to 0$. Consequently, the limit behavior doesn't hold in the same space, see [24].

Let the space $\mathbf{V}^m$ be the completed space of $\mathbf{V}$ for the norm $a^m(\mathbf{v}, \mathbf{v})^{1/2}$, we have the classical result :

*For each $\epsilon > 0$, and $\mathbf{f} \in \mathbf{V}^{m\prime}$ let $\mathbf{u}^{\epsilon}$ be the unique solution of problem (1.7) then*

$$\mathbf{u}^{\epsilon} \longrightarrow \mathbf{u}^{o}\ in\ \mathbf{V}^m\ (strongly),$$

*where $\mathbf{u}^o$ is the solution of :*

$$\left\{ \begin{array}{l} \text{Find a displacement } \mathbf{u}^{0} \in \mathbf{V}^m \text{ such that :} \\ a^m(\mathbf{u}^{0}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{\mathbf{V}^{m\prime}/\mathbf{V}^m}\ \forall \mathbf{v} \in \mathbf{V}^m. \end{array} \right. \tag{10.11}$$

The problem (10.11) is well-posed, but in the space $\mathbf{V}^m$, which strictly contains the space $\mathbf{V}$. In fact, in some cases, the space $\mathbf{V}^m$ can be so large it can even be not contained in the space of distributions.

It is then conveniant to nuance the geometrical rigidity by the notion of *well-inhibition* introduced in [29]:

**Definition 10.1.** *A geometrically rigid surface is said well-inhibited if there exist a positive constant $C$ such that:*

$$\forall \mathbf{u} \in \mathbf{V},\ \|\boldsymbol{\gamma}(\mathbf{u})\|_{L^2} \geq C\,\|\mathbf{u}\|_{\mathbf{V}}\,. \tag{10.12}$$

*A geometrically rigid surface is said non well-inhibited if it is not well-inhibited.*

In other words, a surface is non-well-inhibited, if there exists some sequence of displacements $\mathbf{u}_n$ such that:

$$\left\{ \begin{array}{l} \|\mathbf{u}_n\|_{\mathbf{V}} = 1, \\ \|\boldsymbol{\gamma}(\mathbf{u}_n)\|_{L^2} \longrightarrow 0. \end{array} \right. \tag{10.13}$$

Such displacements are called *pseudo-bendings* and have been introduced in [16].

In the case of a well-inhibited surface, which amounts that the norm $a^m(\mathbf{v}, \mathbf{v})^{1/2}$ is equivalent to the norm $H^1 \times H^1 \times L^2$, the completed space $\mathbf{V}^m$ is contained in $H^1 \times H^1 \times L^2$, see [12].

Actually, the hypothesis of well-inhibition is very restrictive. In fact, pseudo-bendings always exist in hyperbolic or developable surface. Thus, developable or hyperbolic surface are *never* well-inhibited. Moreover, it has been shown in [15] that a surface with an edge is never well inhibited, neither is an elliptic surface with a part of its boundary free of any kinematic conditions. The only known examples of well-inhibited surfaces are ovoids and elliptic surface clamped or fixed all along their boundary, see [15].

With a *not* well-inhibited shell, in some cases, the completed space $\mathbf{V}^m$ can be *not contained* in the distribution space; the shell is then said *sensitive*, in the sense of Lions and Sanchez-Palencia [21], [22].

By duality, it means that the space $\mathcal{D}$ of infinity smooth functions with compact support is *not* contained in the dual space $\mathbf{V}^{m\prime}$.

This means that for a given solution $\mathbf{u}^0$ of problem (10.11) with an external force $\mathbf{f}_0$, the "approximated" problem (10.11) with $\mathbf{f}_0 + \delta\mathbf{f}$ as external force, can have no solution for some $\delta\mathbf{f} \in \mathcal{D}$ but $\delta\mathbf{f} \notin \mathbf{V}^{m'}$; it is a instability phenomenon. In other words, numerical computations are practically unreliable since such studies are subject to approximation (of the data), see [20].

There is an example of sensitive problem given in [21], in the case of an elliptic surface, the boundary of which is constituted by two distinct closed curves (circles like), clamped or fixed on one of these curves, and the other free of conditions. In the framework of section 9 of this paper, we give now a new example of sensitive problem. It has been found in collaboration with E. Sanchez-Palencia.

Let us construct $S$, a surface given by a map $(\Omega, \mathbf{r})$, constituted by three regular *hyperbolic* parts $H$, $H_1$, $H_2$, joined together along curves $\Gamma_1$ and $\Gamma_2$ making two edges with angles different from 0 and $\pi$.

$$\Gamma_1 = H \cap H_1 \text{ and } \Gamma_2 = H \cap H_2$$

We suppose that the two edges are *keeping constant angle* and are *joined together* on a unique point M, as in the figure:

Figure 11: A surface $S$ constituted by 3 hyperbolic surfaces joined together.

We suppose furthermore that in $H$ the PLASP condition is satisfied by the edges

$\Gamma_1$ and $\Gamma_2$. Thus, according to theorem 9.7, the part $H$ is rigid in a neighborhood of the point M.

We impose now the two edges $\Gamma_1$ and $\Gamma_2$ to be transverse to the asymptotic lines of $H$. Consequently, according to remark 9.8, $H$ is entirely rigid if it is included in the determination domain issued from the edges $\Gamma_1$ and $\Gamma_2$.

Last, we suppose that the edges $\Gamma_1$ and $\Gamma_2$ are respectively transversal to the asymptotic lines of $H_1$ and $H_2$. Then, we choose $H_1$ and $H_2$ such that they are respectively contained in the determination, domain issued from the edges $\Gamma_1$ and $\Gamma_2$, in order to have a totally geometrically rigid surface $S$.

**Proposition 10.2.** *Let $S$ be the three hyperbolic parts surface constructed as above, so that it is geometrically rigid. Then the associated problem (10.11) is sensitive.*

**Proof.** We shall exhibit a sequence of displacements of $\mathbf{V}$ such that it is a Cauchy sequence in the space $\mathbf{V}^m$, and therefore have a limit in $\mathbf{V}^m$. We then see that this sequence does not converge in the sense of distributions.

Let the space $\mathbf{V}^l$ be the completed space of $\mathbf{V}$ for the norm associated to the Hilbert space $H^1 \times H^1 \times L^2$. We obviously have

$$\mathbf{V} \subset \mathbf{V}^l \subset \mathbf{V}^m,$$

with dense imbeddings.

It is sufficient then, in the announced process, to take a sequence in $\mathbf{V}^l$ instead of a sequence in $\mathbf{V}$.

On the edge $\Gamma_1$, the continuity of the displacement and the constant angle conditions are

$$\begin{cases} \mathbf{u}^H = \mathbf{u}^{H_1} \\ \delta\theta_1 = 0, \end{cases} \tag{10.14}$$

where $\mathbf{u}^H$ (resp. $\mathbf{u}^{H_1}$) denotes the restriction on $H$ (resp. on $H_1$) of a displacement $\mathbf{u}$ of the surface, and $\delta\theta_1$ is the linearized variation of the angle made at the edge $\Gamma_1$.

In analogous notation, on the edge $\Gamma_2$, the continuity of the displacement and the constant angle conditions are

$$\begin{cases} \mathbf{u}^H = \mathbf{u}^{H_2} \\ \delta\theta_2 = 0. \end{cases} \tag{10.15}$$

Let us choose now a covariant basis which is *orthonormal* along the curve $\Gamma_1$, as done in [15], where the component $u_2$ denotes the component tangent to the curve of edges. The conditions of constant angle (10.14) becomes:

$$\begin{cases} u_2^{H_1} = u_2^H \\ u_3^{H_1} = \frac{1}{2}[\tan(\theta_1).(u_1^{H_1} + u_1^H)] - \cot(\theta_1).(u_1^{H_1} - u_1^H) \\ u_3^H = \frac{1}{2}[-\tan(\theta_1).(u_1^{H_1} + u_1^H)] - \cot(\theta_1).(u_1^{H_1} - u_1^H) \\ u_{3,1}^{H_1} - u_{3,1}^H = b_{1\alpha}^H u_\alpha^H - b_{1\alpha}^{H_1} u_\alpha^H. \end{cases} \tag{10.16}$$

We shall take advantage in the sequel of the fact that in the completion process in $\mathbf{V}^l$ and therefore in $\mathbf{V}^m$, a condition involving the trace of the normal component

$u_3$ does not make sense anymore and therefore *disappear*. In $\mathbf{V}^l$ the only conditions of (10.16) that make sense are:

$$u_2^{H_1} = u_2^H. \qquad (10.17)$$

We recall that a displacement is inextensional if it satisfies the bending system, i.e. if the bilinear form $a^m$ vanishes on it.

Consider now the parts $H$ and $H_2$. According to proposition 7.4 applied on $\Gamma_2$, it is possible to define a non-trivial displacement on $H$ and $H_2$ and satisfying the conditions (10.15).

Let us consider the edge $\Gamma_1$. According to the existence and uniqueness theorem for the Cauchy problem of hyperbolic system, it suffices to impose in $\Gamma_1$:

$$\begin{cases} u_2^{H_1} = u_2^H \\ u_1^{H_1} \text{ is choosen arbitrarily,} \end{cases} \qquad (10.18)$$

to obtain a non trivial inextensional displacement on $H_1$ and satisfying the condition (10.17).

Thus, we have constructed a non-trivial inextensional displacement $\mathbf{u}$ on $S$ (which is not admissible in $\mathbf{V}$!) satisfying to the condition (10.17) on $\Gamma_1$ and (10.15) on $\Gamma_2$.

A priori $u_1^{H_1}$ is unspecified on $\Gamma_1$, but we choose it now such that the jump of the displacement on $\Gamma_1$ is normal to the part $H_1$, as it is possible since the edge makes an angle different from zero and $\pi$, see figure 8.

Figure 12: A transverse view of edge $\Gamma_1$.

Let now K be a non-empty compact included in $H$, as $\mathbf{u}$ is non-trivial, there is a function $\phi \in \mathcal{D}$ such that

$$< \mathbf{u}, \phi >= \int_K \mathbf{u}.\phi \neq 0. \qquad (10.19)$$

Let us perturbate the displacement $\mathbf{u}$ with a displacement field $\mathbf{v}^0$. We choose $\mathbf{v}_0$ such that $\mathbf{v}_0$ vanishes on $H$ and $H_2$ (except on the point M) and such that its tangential components vanish on $H_1$. In other words, we perturbate $\mathbf{u}$ only on $H_1$ by its normal component.

We choose $\mathbf{v}^0$ such that the continuity condition for $\mathbf{u} + \mathbf{v}^0$ is satisfied along $\Gamma_1$. It is possible since we constructed the displacement $\mathbf{u}$ such that its jump on $\Gamma_1$ is normal to $H_1$. In addition, we must verify the fourth condition of (10.16) on $\Gamma_1$ and this may be done easily by choosing appropriately $v_{3,1}^0$.

Thus, the displacement $\mathbf{u}+\mathbf{v}^0$ satisfies the continuity and the constant angle conditions both on the edges $\Gamma_1$ and $\Gamma_2$.

We construct then the sequence of displacements :

$$\mathbf{v}^n = \mathbf{1}_{K_n} \times \mathbf{v}^0, \tag{10.20}$$

where $\mathbf{1}_{K_n}$ is the indicatrix function on the set $K_n$. The sequence of sets $K_n$ are neighborhoods of $\Gamma_1$ included in $H \cup H_1$ such that:

$$K_{n+1} \subset K_n \text{ and } |K_n| = o\left(\frac{1}{n}\right). \tag{10.21}$$

It is clear that each function $\mathbf{u}+\mathbf{v}^n$ satisfies the continuity and constant angle conditions both on the edges $\Gamma_1$ and $\Gamma_2$. We then have constructed a sequence of $\mathbf{V}^l \subset \mathbf{V}^m$.

As $\mathbf{v}^0$ is reduced to its normal component, we have:

$$\|\boldsymbol{\gamma}(\mathbf{u}+\mathbf{v}^n)\|_{L^2(\Omega)} = \|\boldsymbol{\gamma}(\mathbf{v}^n)\|_{L^2(\Omega)} \leq |K_n| \|\boldsymbol{\gamma}(\mathbf{v}^0)\|_{L^2(\Omega)}.$$

Thus,

$$\|\boldsymbol{\gamma}(\mathbf{u}+\mathbf{v}^n)\|_{L^2(\Omega)} \leq o\left(\frac{1}{n}\right). \tag{10.22}$$

In other words, $\mathbf{u}+\mathbf{v}^n$ is a Cauchy sequence in the space $\mathbf{V}^m$. So that, if $\mathbf{V}^m$, is included in the distribution space $\mathcal{D}'$, then, we would have for any function $\psi \in \mathcal{D}$:

$$< \mathbf{u}+\mathbf{v}^n, \psi > \overset{n\to\infty}{\longrightarrow} 0. \tag{10.23}$$

But, for any $n$ sufficiently large, we have:

$$K \cap K_n = \emptyset,$$

thus, for any $n$ sufficiently large, we have:

$$< \mathbf{u}+\mathbf{v}^n, \phi > = < \mathbf{u}+\phi > \neq 0,$$

which is in contradiction with (10.23). □

We give, in this appendix, an existence and uniqueness theorem for a variant of a Goursat problem on a hyperbolic linear P.D.E. system with two variables, used in the proof of theorem 9.6.

But, we first prove a trace lemma (classical) also used in theorem 9.6.

**Lemma A.1.** *Let $u$ be function of $L^2(\Omega)$ such that $u_{,1} \in L^2(\Omega)$ and where $\Omega = [0,T_1] \times [0,T_2]$ is a domain of $R^2$. Let $\Gamma$ be a curve given by a diffeomorphism $\phi$ of $[0,T_1]$ to $[0,T_2]$, such that $\phi(0)=0$:*

$$\Gamma = \{(y^1,y^2) \in \Omega / y^2 = \phi(y^1)\}.$$

*Then, the trace of $u$ on $\Gamma$ make sense in $L^2(\Gamma)$ and we have:*

$$\begin{aligned} \int_0^{T_1} |u(y^1,\phi^{-1}(y^1))|^2 dy^1 &= \int_0^{T_2} |u(\phi(y^2),y^2)|^2 |\phi'(y^2)| dy^2 \\ &\leq |\phi'| \left[\frac{\|u\|_{L^2(\Omega_T)}}{T_2} + \|u_{,1}\|_{L^2(\Omega_T)}\right]. \end{aligned} \tag{.24}$$

**Proof.** In a classical way, we prove the lemma for continuous functions and extend it by continuity.

Let us define the trace of $u$ as the restriction of $u$ on the curve $\Gamma$, usually denoted as $\gamma_0(u)$: with the change of variables $t^1 = \phi(y^2)$, we have:

$$\|\gamma_0(u)\|^2_{L^2(\Gamma)} = \int_0^{T_1} |u(t^1, \phi^{-1}(t^1)|^2 dt^1 = \int_0^{T_2} |u(\phi(y^2), y^2)|^2 |\phi'(y^2| dy^2.$$

But,

$$u(\phi(y^2), y^2) = u(y^1, y^2) - \int_{\phi(y^2)}^{y^1} u_{,1}(t, y^2) dt,$$

consequently, we obtain:

$$\int_0^{T_1} |u(y^1, \phi^{-1}(y^1))|^2 dy^1 \leq |\phi'| \left[ \int_0^{T_2} |u(y^1, y^2)|^2 dy^2 + \int_0^{T_2} \left| \int_{\phi(y^2)}^{y^1} u_{,1}(t, y^2) dt \right|^2 dy^2 \right].$$

Integrating the last relation with respect to the variable $y^1$, we obtain

$$T_1 \int_0^{T_1} |u(y^1, \phi^{-1}(y^1))|^2 dy^1 \leq |\phi'| \left[ \|u\|_{L^2(\Omega_T)} + T_1 \|u_{,1}\|_{L^2(\Omega_T)} \right].$$

In other words, we can extend the trace operator and dividing the last estimation by $T_1$, which proves the lemma. □

**Theorem A.2. (Variant of a Goursat problem)**: *Let $\Omega = [0, T_1] \times [0, T_2]$ be a domain of $R^2$. Let $\Gamma$ be the curve of $\Omega$ defined by a diffeomorphism $\phi$ of $[0, T_1]$ to $[0, T_2]$, such that $\phi(0) = 0$:*

$$\Gamma = \{(y^1, y^2) \in \Omega / y^2 = \phi(y^1)\}.$$

*For any functions $\psi_1 \in L^2[0, T_1]$ and $\psi_2 \in L^2[0, T_2]$ there is a unique solution $(u_1, u_2)$ to the problem:*

$$\left\{ \begin{array}{l} u_{1,1} = a_1^1 u_1 + a_1^2 u_2 \\ u_{2,2} = a_2^1 u_1 + a_2^2 u_2 \end{array} \right. \quad \text{in } \Omega \tag{.25}$$

$$\left\{ \begin{array}{l} u_1(0, y^2) = \psi_2(y^2) \ \forall y^2 \in [0, T_2] \\ u_2(y^1, \phi(y^1)) = \psi_1(y^1) u_1(y^1, \phi(y^1)) \ \forall y^1 \in [0, T_1] \end{array} \right. \tag{.26}$$

*where the coefficients $a_\beta^\alpha$ are smooth functions on $\overline{\Omega}$.*

**Proof.** Let $A$ be the operator defined for any couple of continuous functions on $\Omega$ : $A(u_1, u_2) = (v_1, v_2)$ with:

$$\begin{array}{l} v_1(y^1, y^2) = \int_0^{y^1} a_1^\alpha u_\alpha(t^1, y^2) dt^1 + \psi_2(y^2) \\ v_2(y^1, y^2) = \int_{\phi(y^1)}^{y^2} a_2^\alpha u_\alpha(y^1, t^2) dt^2 + \psi_1(y^1) v_1(y^1, \phi(y^1)). \end{array} \tag{.27}$$

In a classical way, the integral operators in (.27) are extended by continuity into an operator defined from the space $L^2$ into $L^2$.

It is clear that a solution of problem (.25)-(.26) is a fixed point of the operator $A$, and reciprocally.

For any functions $(\tilde{u_1}, \tilde{u_2})$ and $(\bar{u}_1, \bar{u}_2)$, we are going to show the estimate :

$$\|A(\tilde{u_1}, \tilde{u_2}) - A(\bar{u_1}, \bar{u_2})\|_{L^2}) \leq C\|(\bar{u_1}, \bar{u_2}) - (\bar{u_1}, \bar{u_2})\|_{L^2},$$

with a constant $C$ depending on the domain $\Omega$, so that A is a contraction for sufficiently small domain $\Omega$.

Let $(v_1, v_2) = A(\tilde{u_1}, \tilde{u_2}) - A(\bar{u_1}, \bar{u_2})$ and let $(u_1, u_2) = (\tilde{u_1}, \tilde{u_2}) - (\bar{u_1}, \bar{u_2})$, considering the first equation of (.27), there is a positive constant $C$ such that :

$$\|v_1\|_{L^2}^2 \leq CT_1 \left[\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2\right], \tag{.28}$$

where, here and in the sequel, the constant $C$ depends only on coefficients $a_\beta^\alpha$ and on the functions $\psi_1$, $\psi_2$ and $\phi$. In the sequel, $C$ will not be necessarily the same at each occurrence (here, we can take $C = \sup_\Omega(a_\beta^\alpha) + \sup_{[0,T_1]} |\phi^{-1}| \times \sup_{[0,t_2]} |\psi_2|$).

npar In the same way, considering the second equation of (.27), we have

$$\|v_2\|_{L^2}^2 \leq CT_2 \left[\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2\right] + \sup_{[0,T_1]} |\psi_1|^2 \int_0^{T_2} dy^2 \int_0^{T_1} |v_1(y^1, \phi(y^1))|^2 dy^1, \tag{.29}$$

it follows, then, according to lemma 11.1:

$$\|v_2\|_{L^2}^2 \leq CT_2 \left[\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2\right] + C \left[\|v_1\|_{L^2}^2 + T_2\|v_{1,1}\|_{L^2}^2\right]. \tag{.30}$$

But, as $v_{1,1} = a_1^\alpha u_\alpha$, we have the inequality:

$$\|v_{1,1}\|_{L^2}^2 \leq CT_2 \left[\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2\right].$$

Thus, combining the least inequality with (.28) and (.30), we finally have:

$$\|v_2\|_{L^2}^2 \leq CT_2 \left[\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2\right]. \tag{.31}$$

In other words, we have showed that the operator $A$ is k-Lipschitz, with k $\leq C \sup(T_1, T_2)$.

As k $< 1$ for any value of $T_1$ or $T_2$ sufficiently small, the existence and uniqueness of a fixed point (and therefore a solution to the problem (.25)-(.26)) follows from the classical fixed point lemma for contraction operators in Banach spaces. The determination of the solution in the rest of the domain $\Omega$ is then given step by step. □

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